

A Novel Numerical Method for Fuzzy Boundary Value Problems

E Can¹, M A Bayrak² and B Hicdurmaz³

¹ Department of Physical Engineering, Istanbul Medeniyet University, Istanbul, Turkey

² Department of Mathematics, Kocaeli University, Kocaeli, Turkey

³ Department of Mathematics, Istanbul Medeniyet University, Istanbul, Turkey

E-mail: emine.can@medeniyet.edu.tr

Abstract. In the present paper, a new numerical method is proposed for solving fuzzy differential equations which are utilized for the modeling problems in science and engineering. Fuzzy approach is selected due to its important applications on processing uncertainty or subjective information for mathematical models of physical problems. A second-order fuzzy linear boundary value problem is considered in particular due to its important applications in physics. Moreover, numerical experiments are presented to show the effectiveness of the proposed numerical method on specific physical problems such as heat conduction in an infinite plate and a fin.

1. Introduction

Fuzzy differential equations (FDEs) are utilized as a contribution to the improvement of the modelling problems in science and engineering. The difference of fuzzy approach is its suitable structure for uncertainty and subjective information on mathematical models of real world problems. As it is known to all, the mathematical models are used in various problems like population, quantum optics, gravity, control design, medicine etc. In most cases, physical problems are modelled by a boundary value problem. So, considering the boundary value problem with a fuzzy approach, gives us the fuzzy boundary value problems. A fuzzy boundary value problem may have vagueness in each part of the equation like initial condition, boundary condition etc. Due to its important applications, the fuzzy boundary value problems will be considered in the present talk. Then, applications of fuzzy boundary value problems on the flow of heat in an infinite plate and a fin will be presented.

According to the oldest sources on fuzzy derivatives, fuzzy differentiation started with the introduction of some basic properties of fuzzy set theory by Chang and Zadeh in their study [1]. Then, the paper of Dubois and Prade contributed to this definition in their paper [2].

In the literature, fuzzy differential equations have been studied by several different ways. The first and the most popular approach is using the Hukuhara differentiability for fuzzy-valued functions [3]. Unfortunately, this approach has a drawback: the solution becomes fuzzier as time goes by. Hence, the fuzzy solution behaves quite differently from the crisp solution. As an alternative to this definition, Seikkala [4] introduced the notion of fuzzy derivative as an extension of the Hukuhara derivative and fuzzy integral. To solve some drawbacks of generalization of fuzzy differential equations, Hllermeier [5] interpreted FDE as a family of differential inclusions.



The main shortcoming of using differential inclusions is that we do not have a derivative of a fuzzy-valued function. Then, the strongly generalized differentiability was introduced in [6] and studied in [7, 8]. But, the uniqueness of solutions for fuzzy differential equations is a problem under strongly generalized differentiability. Hence, we will recall and use H-differentiability concept which is preferred extensively [9-17] in the present paper.

Most of the problems in science and engineering require the solutions of a fuzzy differential equation (FDE) which are satisfied in fuzzy boundary values. A. Khastan and J. J. Nieto interpret a two-point boundary value problem for a second order fuzzy differential equation by using the generalized differentiability. So, the method of paper [18] is the selected approach of authors for handling the fuzzy differential equations in the present study.

On the other hand, numerical solutions of fuzzy differential equations draw great interest due to its important applications in physical problems. Numerical solution of a FDE is obtained now in a natural way, by extending the existing classical methods to the fuzzy case. Some numerical methods for FDE under the Hukuhara differentiability concept such as the fuzzy Adams-Bashforth, Euler, Adams-Moulton, Predictor-Corrector, Nystrom methods on FDEs appear in the literature [19-22]. Numerical methods for boundary value problems for linear second order FDEs are also considered in several studies [23, 24].

In the present study, the main aim is to solve a second-order fuzzy linear boundary value problem with a novel numerical method which is based on chasing method which came from the studies which are printed in [25] and [26]. Chasing method is based on the idea to establish a set of auxiliary problems that can be solved to find initial conditions at one of the boundaries. After the determination of initial conditions, the usual methods for solving initial boundary value problems can be applied.

The paper is organized as follows: In Section 2, the necessary preliminary information on fuzzy set theory and H-differentiability is given. In Section 3, fuzzy boundary value problems under generalized differentiability are presented and investigated. A new method for solving considered problem is introduced in Section 4. In Section 5, some boundary value problems for fuzzy heat equations are solved to illustrate effectiveness of the proposed method on several cases for physical applications.

2. Preliminaries

In the present section, some definitions and necessary notations on fuzzy set theory and fuzzy differentiability will be introduced.

2.1. Fuzzy set theory

Definition 2.1. Let X be a nonempty set. A fuzzy set u in X is characterized by its membership function $u : X \rightarrow [0, 1]$. Thus, $u(x)$ is interpreted as the degree of membership of an element x in the fuzzy set u for each $x \in X$.

Let us denote by $\mathcal{F}$ the class of fuzzy subsets of the real axis (i.e., $u : R \rightarrow [0, 1]$) satisfying the following properties:

- (i) u is normal, that is, there exists $x_0 \in R$ such that $u(x_0) = 1$.
- (ii) u is a convex fuzzy set (i.e. $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$, for all $\lambda \in [0, 1], x, y \in R$).
- (iii) u is upper semicontinuous on R .
- (iv) $cl\{x \in R | u(x) > 0\}$ is compact where cl denotes the closure of a subset.

Then $\mathcal{F}$ is called the space of fuzzy numbers. For each $\alpha \in (0, 1]$ the α -level set $[u]^\alpha$ of a fuzzy set u is the subset of points $x \in R$ with

$$[u]^\alpha = [\underline{u}^\alpha, \bar{u}^\alpha] \quad (1)$$

where $\underline{u}$ and $\bar{u}$ are called lower and upper branch of u respectively.
For $u \in E$ we define the length of u as:

$$\text{len}(u) = \sup_{\alpha} (\bar{u}^{\alpha} - \underline{u}^{\alpha}). \quad (2)$$

A fuzzy number in parametric form is presented by an ordered pair of functions $(\underline{u}^{\alpha}, \bar{u}^{\alpha}), 0 \leq \alpha \leq 1$, satisfying the following properties:

(i) $\underline{u}^{\alpha}$ is a bounded nondecreasing left-continuous function of α over $(0, 1]$ and right continuous for $\alpha = 0$.

(ii) $\bar{u}^{\alpha}$ is a bounded nonincreasing left-continuous function of α over $(0, 1]$ and right continuous for $\alpha = 0$.

(iii) $\underline{u}^{\alpha} \leq \bar{u}^{\alpha}, 0 \leq \alpha \leq 1$.

For $u, v \in \mathcal{F}$ and $\lambda \in R$, the sum $u + v$ and the product λu are defined by $[u + v]^{\alpha} = [u]^{\alpha} + [v]^{\alpha}$ and $[\lambda u]^{\alpha} = \lambda[u]^{\alpha}$, for all $\alpha \in [0, 1]$, where $[u]^{\alpha} + [v]^{\alpha}$ means the usual addition of two intervals (subsets) of R and $\lambda[u]^{\alpha}$ means the usual product between a scalar and a subset of R . The metric on E is defined by the equation

$$D(u, v) = \sup_{\alpha \in [0, 1]} \max\{|\underline{u}^{\alpha} - \underline{v}^{\alpha}|, |\bar{u}^{\alpha} - \bar{v}^{\alpha}|\}. \quad (3)$$

Here, D is a Hausdorff distance of two intervals $[u]^{\alpha}$ and $[v]^{\alpha}$.

2.2. Hukuhara differentiability

Definition 2.2. Let $x, y \in \mathcal{F}$. If there exists $z \in \mathcal{F}$ such that $x = y + z$, then z is called the H -difference of x, y and it is denoted by $x \ominus y$.

Definition 2.3. Let $I = (0, l)$ and $f : I \rightarrow \mathcal{F}$ is a fuzzy function. We say that f is differentiable at $t_0 \in I$ if there exists an element $f'(t_0) \in \mathcal{F}$ such that the limits

$$\lim_{h \rightarrow 0^+} \frac{f(t_0 + h) \ominus f(t_0)}{h}, \lim_{h \rightarrow 0^+} \frac{f(t_0) \ominus f(t_0 - h)}{h}$$

exist and are equal to $f'(t_0)$. Here, the limits are taken in the metric space $(\mathcal{F}, D)$.

Definition 2.4. Let $f : I \rightarrow \mathcal{F}$ and fix to $t_0 \in I$. We say that

(i) f is (1) differentiable at t_0 , if there exists an element $f'(t_0) \in \mathcal{F}$ such that for all $h > 0$ sufficiently near to 0, there exist $f(t_0 + h) \ominus f(t_0)$, $f(t_0) \ominus f(t_0 - h)$, and the limits (in the metric D) satisfy:

$$\lim_{h \rightarrow 0^+} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \rightarrow 0^+} \frac{f(t_0) \ominus f(t_0 - h)}{h} = F'(t_0) \quad (4)$$

(ii) f is (2) differentiable at t_0 , if there exists an element $f'(t_0) \in \mathcal{F}$ such that for all $h > 0$ sufficiently near to 0, there exist $f(t_0 + h) \ominus f(t_0)$, $f(t_0) \ominus f(t_0 - h)$, and the limits (in the metric D) satisfy:

$$\lim_{h \rightarrow 0^-} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \rightarrow 0^-} \frac{f(t_0) \ominus f(t_0 - h)}{h} = F'(t_0) \quad (5)$$

If f is (n) -differentiable at t_0 , we denote its first derivatives by $D_n^{(1)} f(t_0)$, for $n = 1, 2$.

It is obvious that Hukuhara differentiable function has an increasing length of support. If the function doesn't have these properties then this function is not H -differentiable. To avoid this difficulty the authors in [11] introduced the following more general definition of derivative for fuzzy valued functions:

Theorem 2.1. Let $f : I \rightarrow \mathcal{F}$ be fuzzy function, where $[f(t)]^\alpha = [\underline{f}_\alpha(t), \bar{f}_\alpha(t)]$ for each $\alpha \in [0, 1]$.

(i) If f is (1) differentiable in the first form, then $\underline{f}_\alpha$ and $\bar{f}_\alpha$ are differentiable functions and $[D_1^1 f(t)]^\alpha = [\underline{f}'_\alpha(t), \bar{f}'_\alpha(t)]$.

(ii) If f is (2) differentiable in the first form, then $\underline{f}_\alpha$ and $\bar{f}_\alpha$ are differentiable functions and $[D_2^1 f(t)]^\alpha = [\bar{f}'_\alpha(t), \underline{f}'_\alpha(t)]$.

Proof of Theorem 2.1 is given in [17].

The following theorem gives details on the generalized differentiability:

Theorem 2.2. Let $D_1^1 f : I \rightarrow \mathcal{F}$ or $D_2^1 f : I \rightarrow \mathcal{F}$ be fuzzy functions, where $[f(t)]^\alpha = [\underline{f}_\alpha(t), \bar{f}_\alpha(t)]$ for $\forall \alpha \in [0, 1]$. Then

(i) If $D_1^1 f$ is (1) differentiable, then $\underline{f}'^\alpha$ and $\bar{f}'^\alpha$ are differentiable functions and $[D_{1,1}^2 f(t)]^\alpha = [\underline{f}''_\alpha(t), \bar{f}''_\alpha(t)]$.

(ii) If $D_1^1 f$ is (2) differentiable, then $\underline{f}'^\alpha$ and $\bar{f}'^\alpha$ are differentiable functions and $[D_{1,2}^2 f(t)]^\alpha = [\bar{f}''_\alpha(t), \underline{f}''_\alpha(t)]$.

(iii) If $D_2^1 f$ is (1) differentiable, then $\underline{f}'^\alpha$ and $\bar{f}'^\alpha$ are differentiable functions and $[D_{2,1}^2 f(t)]^\alpha = [\bar{f}''_\alpha(t), \underline{f}''_\alpha(t)]$.

(iv) If $D_2^1 f$ is (2) differentiable, then $\underline{f}'^\alpha$ and $\bar{f}'^\alpha$ are differentiable functions and $[D_{2,2}^2 f(t)]^\alpha = [\underline{f}''_\alpha(t), \bar{f}''_\alpha(t)]$.

Proof of Theorem 2.2 is presented in detail in [17].

3. Fuzzy Boundary Value Problem

We consider the two-point fuzzy boundary value problem which is defined as follows:

$$y''(t) = F(t, y(t), y'(t)), \quad t \in J = [0, T] \quad (6)$$

with boundary conditions $y(0) = \gamma_\alpha$, $y'(T) = \lambda_\alpha$ where $y(t)$ is an unknown fuzzy function of crisp variable t , $f : J \times R^2 \rightarrow R$ is continuous and $\phi_1, \phi_2 \in R$.

Let $y(t)$ be a fuzzy-valued function on J represented by $[\underline{y}(t; r), \bar{y}(t; r)]$. Let $y(t)$, $y'(t)$ are H -differentiable on J , then problem (6) can be transformed to the following BVPs systems by the method of Khastan and Nieto [17]: If $y(t)$ and $y'(t)$ are 1-differentiable on J , then y is called (1, 1)-solution for problem (6), and obtained from

$$\begin{aligned} \underline{y}''(x, \alpha) &= F(x, \underline{y}(x, \alpha), \underline{y}'(x, \alpha)) \\ \bar{y}''(x, \alpha) &= F(x, \bar{y}(x, \alpha), \bar{y}'(x, \alpha)) \\ \underline{y}(a, \alpha) &= \underline{\gamma}_\alpha, \quad \bar{y}(a, \alpha) = \bar{\gamma}_\alpha \\ \underline{y}'(b, \alpha) &= \underline{\lambda}_\alpha, \quad \bar{y}'(b, \alpha) = \bar{\lambda}_\alpha. \end{aligned}$$

If $y(t)$ is 1-differentiable and $y'(t)$ is 2-differentiable on J , then y is called (1, 2)-solution for problem (6), and obtained from

$$\begin{aligned} \underline{y}''(x, \alpha) &= F(x, \bar{y}(x, \alpha), \bar{y}'(x, \alpha)) \\ \bar{y}''(x, \alpha) &= F(x, \underline{y}(x, \alpha), \underline{y}'(x, \alpha)) \\ \underline{y}(a, \alpha) &= \underline{\gamma}_\alpha, \quad \bar{y}(a, \alpha) = \bar{\gamma}_\alpha \\ \underline{y}'(b, \alpha) &= \underline{\lambda}_\alpha, \quad \bar{y}'(b, \alpha) = \bar{\lambda}_\alpha. \end{aligned}$$

If $y(t)$ is 2-differentiable and $y'(t)$ is 1-differentiable on J , then y is called (2, 1)-solution for problem (6), and obtained from

$$\begin{aligned}
y''(x, \alpha) &= F(x, \bar{y}(x, \alpha), y'(x, \alpha)) \\
\bar{y}''(x, \alpha) &= F(x, y(x, \alpha), \bar{y}'(x, \alpha)) \\
\underline{y}(a, \alpha) &= \underline{\gamma}_\alpha, \bar{y}(a, \alpha) = \bar{\gamma}_\alpha \\
\underline{y}'(b, \alpha) &= \underline{\lambda}_\alpha, \bar{y}'(b, \alpha) = \bar{\lambda}_\alpha.
\end{aligned}$$

If $y(t)$ and $y'(t)$ are 2-differentiable on J , then y is called (2, 2)-solution for problem (6), and obtained from

$$\begin{aligned}
y''(x, \alpha) &= F(x, y(x, \alpha), \bar{y}'(x, \alpha)) \\
\bar{y}''(x, \alpha) &= F(x, \bar{y}(x, \alpha), \underline{y}'(x, \alpha)) \\
\underline{y}(a, \alpha) &= \underline{\gamma}_\alpha, \bar{y}(a, \alpha) = \bar{\gamma}_\alpha \\
\underline{y}'(b, \alpha) &= \bar{\lambda}_\alpha, \bar{y}'(b, \alpha) = \underline{\lambda}_\alpha.
\end{aligned}$$

Our strategy for solving problem (6) is based on the choice of derivative type in the fuzzy boundary value problem. According to the chosen derivative type, the fuzzy boundary value problem is converted into the corresponding system of boundary value problems. Finally, we consider the domain where the solution and its derivatives have valid level sets according to the differentiability type.

4. Chasing method

Chasing method for solving boundary value problems was developed by Gel'fand and Lokutsiyevskii and it was introduced in detail by I.S.Berzin and N.F.Zhidkov [22, 23]. The method is based on the idea to establish a set of auxiliary problems that can be solved to find initial conditions at one of the boundaries. After the determination of initial conditions, the usual methods for solving initial boundary value problems can be applied.

In the present section, we will present the application of chasing method on a linear fuzzy boundary value problem. Firstly, the fuzzy second order differential equation

$$y''(x, \alpha) = p(x)y(x, \alpha) + q(x) \quad (7)$$

where $p(x), q(x)$ are known continuous functions is considered. The boundary conditions are given as

$$y'(a, \alpha) = \alpha_{00}y(a, \alpha) + \alpha_{10} \quad (8)$$

$$y'(b, \alpha) = \beta_{00}y(b, \alpha) + \beta_{10} \quad (9)$$

where $\alpha_{00}, \alpha_{10}, \beta_{00}, \beta_{10}$ are known constants.

We now consider a linear fuzzy first-order differential equation

$$y'(x, \alpha) = \alpha_0(x)y(x, \alpha) + \alpha_1(x) \quad (10)$$

and choose $\alpha_0(x)$ and $\alpha_1(x)$ so that y still satisfies equation (7). Differentiating equation (10) with respect to x and replacing $y'(x, \alpha)$ by the right-hand side of equation (10), we get

$$y''(x, \alpha) = (\alpha'_0(x) + [\alpha_0(x)]^2)y(x, \alpha) + (\alpha'_1(x) + \alpha_0(x)\alpha_1(x)). \quad (11)$$

From equation (7), it is seen that the following equations must be satisfied

$$\alpha'_0(x) + [\alpha_0(x)]^2 = p(x) \quad , \quad \alpha_0(a) = \alpha_{00} \quad (12)$$

$$\alpha'_1(x) + \alpha_0(x)\alpha_1(x) = q(x) \quad , \quad \alpha_1(a) = \alpha_{10} \quad (13)$$

Then, the two quantities $\alpha_0(b)$ and $\alpha_1(b)$ are obtained. From equation (10), we write

$$y'(b, \alpha) = \alpha_0(b)y(b, \alpha) + \alpha_1(b) \quad (14)$$

and equation (9) gives

$$y'(b, \alpha) = \beta_{00}y(b, \alpha) + \beta_{10}. \quad (15)$$

Since $\alpha_0(b)$ and $\alpha_1(b)$ are known quantities, equations (14) and (15) can be solved for

$$y(b, \alpha) = \frac{\beta_{10} - \alpha_1(b)}{\alpha_0(b) - \beta_{00}} \quad (16)$$

$$y'(b, \alpha) = \frac{\beta_{00}\alpha_1(b) - \beta_{10}\alpha_0(b)}{\beta_{00} - \alpha_0(b)}. \quad (17)$$

Thus, equation (7) is transformed into an initial value problem. Boundary conditions of problem (6) can be converted into the form of conditions (8) and (9). Also, problem (7) is a special case of problem (6). This allows us the combination of chasing method with fuzzy methodology on two point boundary value problems. The combination of chasing method with fuzzy methodology gives a new numerical method which is very effective on finding approximate solutions of different types of fuzzy boundary value problems. It is applicable to fuzzy boundary value problems with different boundary conditions. The effectiveness of the method on numerical solutions of fuzzy boundary value problems will be illustrated by the applications on physical problems in the next section.

5. Solutions for heat transfer problems

Heat transfer problems form an important part of the physical heat problems. Applications of them in engineering are including but not limited to resistance, cooling technologies and architecture. Due to its wide application area, we will present the application of the new constructed method on the following heat transfer problems.

5.1. Heat conduction in an infinite plate with heat generation

In the present section, we firstly consider an infinite flat plate which separates a fluid at a temperature of T_∞ . Here, heat is generated within the plate at a constant rate, q_s .

To formulate the problem, the first law of thermodynamics can be written as

$$\{Aq_x\} + \{q_s A dx\} = \{A(q_x + dq_x)\}$$

where A is the area.

Simplifying the above equation and introducing the Fourier law of conduction,

$$q_x = -k \frac{dT}{dx}$$

the first law of thermodynamics becomes

$$\frac{d^2T}{dx^2} + \frac{q_s}{k} = 0 \quad (18)$$

where total differentiation is used since the temperature is a function of x only.

Since the temperature gradient must be zero at the plane midway between the two surfaces of the plate, it can be formulated as

$$\frac{dT(0)}{dx} = 0.$$

The second boundary condition is written in the usual way when the solid surface is in contact with a fluid at a different temperature with negligible radiation:

$$-k \frac{dT(l)}{dx} = \bar{h}[T(l) - T_\infty]$$

where k is the heat conductivity, $\bar{h}$ is the convective heat transfer coefficient, T_∞ is the temperature of the fluid away from the surface of the plate, and l is half the thickness of the plate.

Introducing the dimensionless quantities

$$\xi = \frac{x}{l}, \quad \theta = \frac{T - T_\infty}{q_s l^2 / k},$$

equation (17) becomes

$$\theta''(\xi, \alpha) = -1 \quad (19)$$

subject to the boundary conditions

$$\theta'(0, \alpha) = 0 \quad (20)$$

$$-\theta'(1, \alpha) = N_{bi}\theta(1, \alpha) \quad (21)$$

where N_{bi} is the Biot number.

The exact solution of equation (18) is

$$\underline{\theta}(\xi, \alpha) = (\alpha - 1)\left[\frac{1}{2}(1 - \xi^2) + \frac{1}{N_{bi}}\right], \quad (22)$$

$$\bar{\theta}(\xi, \alpha) = (1 - \alpha)\left[\frac{1}{2}(1 - \xi^2) + \frac{1}{N_{bi}}\right]. \quad (23)$$

In Fig. 1 and 2, the exact solution and solutions of (1,1) and (1,2) systems are presented and for (2,1) and (2,2) solutions, we get the results that illustrated in Fig. 3 and 4 with $\alpha = 0$ and $N_{bi} = 0.5, 1.5, 3$.

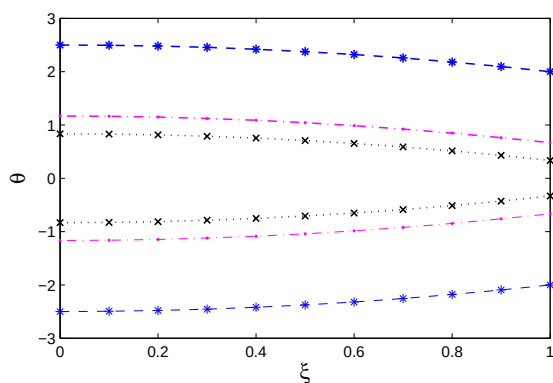


Figure 1. Solution (1,1) of problem (18) for $N_{bi} = 0.5(- * -), 1.5(- \cdot -), 3(\cdot \times \cdot)$.

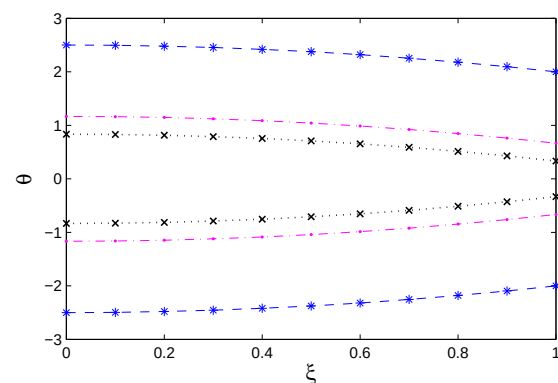


Figure 2. Solution (1,2) of problem (18) for $N_{bi} = 0.5(- * -), 1.5(- \cdot -), 3(\cdot \times \cdot)$.

5.2. Heat transfer in a fin

Here, it is a good approximation to assume that the temperature over every cross section perpendicular to the axis of the fin is uniform in the design of fins. The temperature at the base ($x = 0$) of the fin is known as T_s . If we consider an infinitesimal length dx of the fin, taken as the control volume, conservation of energy can be written as

$$\{Aq_x\} = \{A(q_x + dq_x)\} + \{\bar{h}(Pdx)(T - T_\infty)\}$$

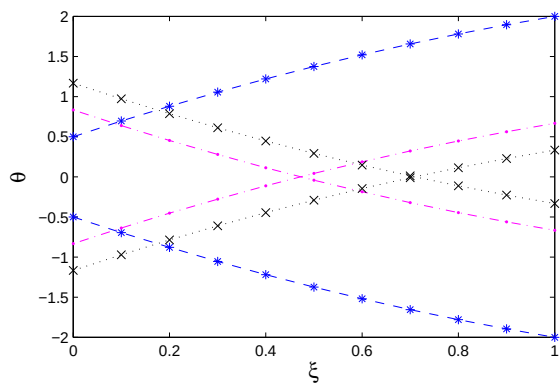


Figure 3. Solution (2,1) of problem (18) for $N_{bi} = 0.5(- * -), 1.5(- \cdot -), 3(\cdot \times \cdot)$.

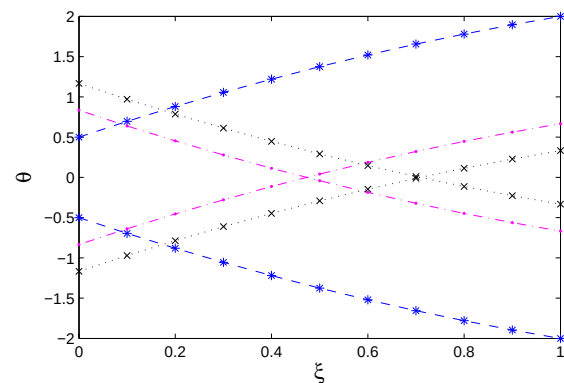


Figure 4. Solution (2,2) of problem (18) for $N_{bi} = 0.5(- * -), 1.5(- \cdot -), 3(\cdot \times \cdot)$.

where A and P are the cross-sectional area and the perimeter, respectively, and $\bar{h}$ is the convective heat transfer coefficient.

Simplifying the above equation and introducing the Fourier law of heating,

$$q_x = -k \frac{dT}{dx}$$

we get

$$\frac{d^2T}{dx^2} = m^2(T - T_\infty), \quad m = \frac{\bar{h}P}{kA}. \quad (24)$$

The temperature at the base ($x = 0$) of the fin is known as T_s , i.e. $T = T_s$. On the other surface, $x = L$, the plate is exposed to air and the only mode of heat transfer is by convection. Equating the conduction out of the solid to the convection into the fluid, we get

$$-k \frac{dT(L)}{dx} = \bar{h}[T(L) - T_\infty].$$

If the following dimensionless quantities are introduced

$$\xi = \frac{x}{l}, \quad \theta = \frac{T - T_\infty}{T_s - T_\infty},$$

equation (23) becomes

$$\theta''(\xi, \alpha) = \beta \theta(\xi, \alpha) \quad (25)$$

subject to the boundary conditions

$$\theta(0, \alpha) = 1 \quad (26)$$

$$-\theta'(1, \alpha) = N_{bi} \theta(1, \alpha) \quad (27)$$

where

$$\beta = m^2 L^2, \quad N_{bi} = \frac{\bar{h}L}{k}.$$

The exact solution of equation (24) is

$$\theta(\xi, \alpha) = (\alpha - 1) \left[\frac{\sqrt{\beta} \cosh \sqrt{\beta}(1 - \xi) + N_{bi} \sinh \sqrt{\beta}(1 - \xi)}{\sqrt{\beta} \cosh \sqrt{\beta} + N_{bi} \sinh \sqrt{\beta}} \right], \quad (28)$$

$$\bar{\theta}(\xi, \alpha) = (1 - \alpha) \left[\frac{\sqrt{\beta} \cosh \sqrt{\beta}(1 - \xi) + N_{bi} \sinh \sqrt{\beta}(1 - \xi)}{\sqrt{\beta} \cosh \sqrt{\beta} + N_{bi} \sinh \sqrt{\beta}} \right]. \quad (29)$$

In Fig. 5 and 6, the exact solution and solutions of (1,1) and (1,2) systems are presented for $\alpha = 0$, $\beta = 2$ and $N_{bi} = 0.5, 1.5, 3$. For (2,1) and (2,2) solutions, we get the results that illustrated in Fig. 7 and 8 for $\alpha = 0$, $\beta = 1$ and $N_{bi} = 0.5, 1.5, 3$.

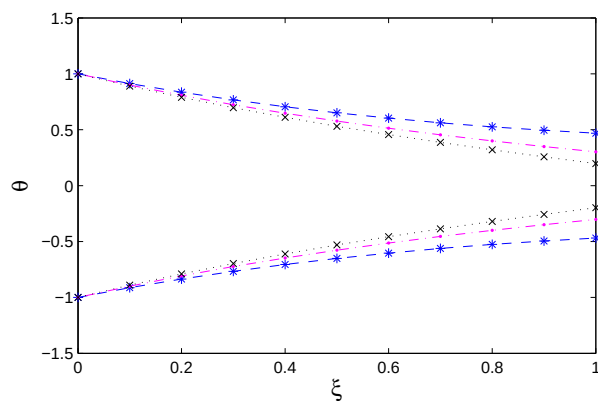


Figure 5. Solution (1,1) of problem (24) for $N_{bi} = 0.5(- * -), 1.5(- \cdot -), 3(\cdot \times \cdot)$.

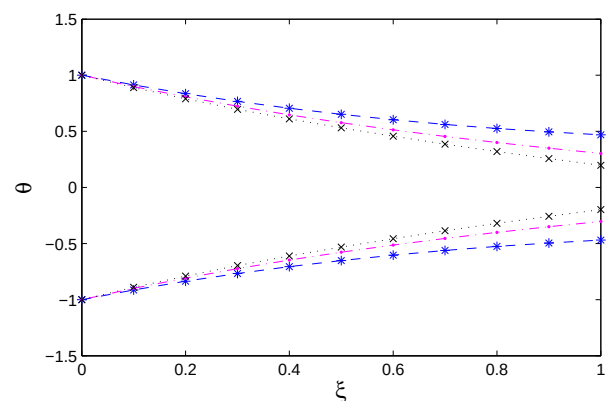


Figure 6. Solution (1,2) of problem (24) for $N_{bi} = 0.5(- * -), 1.5(- \cdot -), 3(\cdot \times \cdot)$.

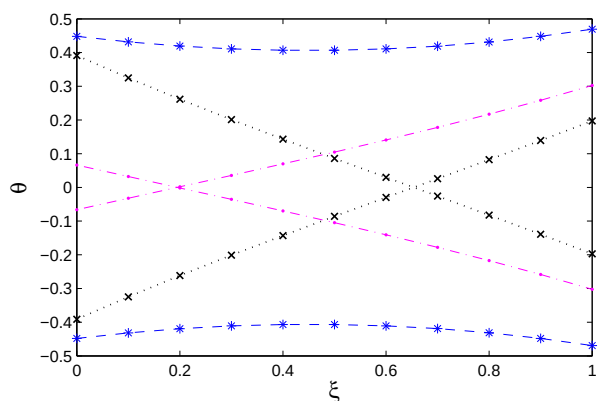


Figure 7. Solution (2,1) of problem (24) for $N_{bi} = 0.5(- * -), 1.5(- \cdot -), 3(\cdot \times \cdot)$.

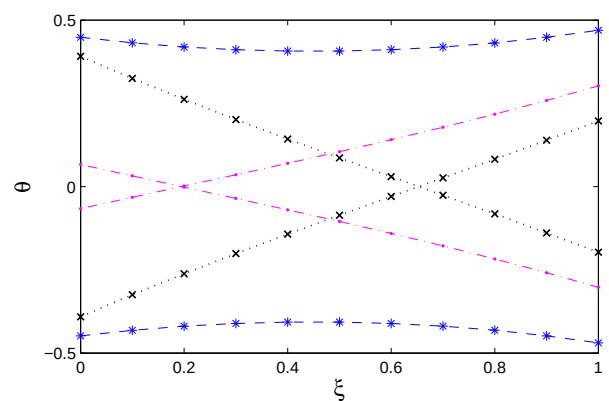


Figure 8. Solution (2,2) of problem (24) for $N_{bi} = 0.5(- * -), 1.5(- \cdot -), 3(\cdot \times \cdot)$.

6. Conclusions

In the paper, a new numerical method has been introduced for fuzzy boundary value problems under generalized differentiability. This method offers a very interesting alternative for the conversion from a fuzzy boundary value problem to a fuzzy initial value problem. Another advantage of this method is the property that under certain conditions the solutions of the equations of forward and backward chasing methods increase slowly with independent variable. The proposed method can be applied to many physical problems with uncertainty and vagueness. In this paper, effectiveness of the proposed method is tested on heat transfer problems in an infinite plate and a fin.

References

- [1] Zadeh L A 1975 *Inform. Sci.* **8** (1975) 199-251
- [2] Dubois D and Prade H 1982 *Fuzzy Sets and Syst.* **8** 225-233
- [3] Kaleva O and Thomas F 1987 *Fuzzy Sets and Syst.* **24** 301-317
- [4] Seikkala S 1987 *Fuzzy Sets and Syst.* **24** 319-330
- [5] Hllermeier E 1997 *Int. J. Uncertain Fuzz.* **5** 117-137
- [6] Stefanini L and Bede B 2009 *Nonlin. Anal.: Theory, Meth. Appl.* **71** 1311-1328
- [7] Bede B, Rudas I J and Bencsik A L 2007 *Inf.Sci.* **177** 1648-1662
- [8] Chalco-Cano Y and Roman-Flores H 2008 *Chaos and Sol. Fract.* **38** 112-119
- [9] Chen M, Li D and Xue X 2011 *Fuzzy Sets and Syst.* **162** 67-78
- [10] Diamond P 2000 *IEEE Trans. on Fuzzy Syst.* **8** 583-590
- [11] Buckley J J and Thomas F 2000 *Fuzzy Sets and Syst.* **110** 43-54
- [12] Diamond P and Watson P 2000 *Appl. Math. Lett.* **13** 31-35
- [13] O'Reagan D, Lakshmikantham V and Nieto J 2003 *Nonlin. Anal.* **54** 405-415
- [14] Ma M, Friedman M and Kandel A 1999 *Fuzzy Sets and Syst.* **105** 133-138
- [15] Abbasbandy S and Allahviranloo T 2004 *Nonlin. Studies* **11** 117-129
- [16] Puri M L, Ralescu D A 1983 *J. of Math. Anal. and Appl.* **91** 552-558
- [17] Golayoglu A F and Koroglu C 2011 *CMES-Comput. Model. in Eng. Sci.* **75** 189-204
- [18] Khastan A and Nieto J J 2010 *Nonlin. Anal.* **72** 3583-3593
- [19] Palligkinis S C, Papageorgiou G and Famelis I T 2009 *Appl. Math. and Comput.* **209** 97-105
- [20] Allahviranloo T, Ahmadi N and Ahmadi E, 2007 *Inf. Sci.* **177** 1633-1647
- [21] Friedman M, Ma M and Kandel A 1999 *Fuzzy Sets and Syst.* **105** 35-48
- [22] Khastan A and Ivaz K 2009 *Chaos and Sol. Fract.* **41** 859-868
- [23] Allahviranloo T and Khalilpour K 2011 *World Appl. Sci. J.* **13** 2137-2147
- [24] Khastan A, Bahrami F and Ivaz K 2009 *Boun. Val. Prob.* **395714** 1
- [25] Berezin I S and Zhidkov N P 1965, *Computing methods* vol 2 (Oxford: Pergamon Press)
- [26] Na T Y 1979 *Computational methods in Engineering Boundary Value Problems* (New York: Academic Press)