# Theorems On The Subsequential Convergence

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**Abstract.** In this study we obtain some sufficient conditions under which subsequential convergence of a sequence of real numbers follows from its boundedness. Eventually, we obtain crucial information about the subsequential behavior of sequences.
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## INTRODUCTION

It is well known that convergence of a sequence $\{s_n\}$ of real numbers implies its boundedness, yet the converse is not necessarily true is clear from the example of $\{\sin(n\pi/2)\}$. Since boundedness is a necessary condition for convergence of $\{s_n\}$, we put the following question: Under which conditions we get information on the convergence behavior of bounded sequences. In the case where $\{s_n\}$ is monotonic and bounded, we have its convergence. On the other hand, Bolzano-Weierstrass theorem states that every bounded sequence has at least one accumulation point. However, there are some bounded sequences such as $\{\sin(\log n)\}$ whose accumulation points lie on a finite interval and all points in this interval are accumulation points of the sequence. In this case we just have convergence of some subsequences of $\{s_n\}$. Motivated by this idea, Stanojević [1] defined a new kind of convergence as follows.

**Definition 1** A sequence $\{s_n\}$ is said to be subsequentially convergent if there exists a finite interval $I$ such that all accumulation points of the sequence $\{s_n\}$ are in $I$ and every point of $I$ is an accumulation point of $\{s_n\}$.

Throughout this paper, we adopt the following familiar conventions:

(i) $a_n = o(b_n)$ means $a_n/b_n \to 0$ as $n \to \infty$,
(ii) $a_n = O(b_n)$ means $|a_n| \le Hb_n$ for sufficiently large $n$, where $H$ is a positive constant,
(iii) $a_n \sim b_n$ means $a_n/b_n \to 1$ as $n \to \infty$.

Note that every convergent sequence is subsequentially convergent. Further, it is obvious that subsequential convergence implies boundedness. But the converse is not always valid, provided by the example $\{(-1)^n\}$. The first theorem which reveals that the converse is valid under certain conditions was obtained by Dik [2] as stated below.

**Theorem 1** If $\{s_n\}$ is a bounded sequence such that $\Delta s_n = o(1)$ as $n \to \infty$, then $\{s_n\}$ is subsequentially convergent.

Using Theorem 1 we can easily show that $s_n = \{\sin(\log n)\}$ is subsequentially convergent. Indeed, since $\{s_n\}$ is bounded and

$$\begin{aligned} |\Delta s_n| = |\Delta \sin(\log n)| &= |\sin(\log n) - \sin(\log(n-1))| \\ &\le |\log n - \log(n-1)| = o(1),\ n \to \infty, \end{aligned}$$

$\{s_n\}$ is subsequentially convergent by Theorem 1.

Subsequential convergence was studied in a number of papers such as Çanak and Totur [3, 4], Dik [2], Dik et al. [5] and Sezer and Çanak [6]. In this paper we investigate conditions under which subsequential convergence of $\{s_n\}$ follows from its boundedness.

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## PRELIMINARIES

In this section, we present some fundamental definitions, identities and lemmas which will be needed in the sequel.

The logarithmic mean of $\{s_n\}$ is defined by

$$t_n^{(1)}(s) = \frac{1}{\ell_n}\sum_{k=0}^{n}\frac{s_k}{k+1}, \quad \text{where} \quad \ell_n = \sum_{k=0}^{n}\frac{1}{k+1} \sim \log n, \quad n = 0, 1, 2, \ldots . \tag{1}$$

**Definition 1** A sequence $\{s_n\}$ is said to be summable to a finite number $L$ by the logarithmic mean method $(\ell, 1)$ if $\lim_{n\to\infty} t_n^{(1)}(s) = L$. In this case, we write $s_n \to \xi(\ell, 1)$.

The difference between a sequence $s_n$ and its logarithmic mean $t_n^{(1)}(s)$, that is known as the logarithmic Kronecker identity (see [7]) is given by

$$s_n - t_n^{(1)}(s) = v_n^{(0)}(\Delta s) \tag{2}$$

where $v_n^{(0)}(\Delta s) = \dfrac{1}{\ell_n}\displaystyle\sum_{k=1}^{n}\ell_{k-1}\Delta s_k$ and $\Delta s_n = s_n - s_{n-1}$ with $s_{-1} = 0$.

Since identity (2) can be rewritten as

$$s_n = v_n^{(0)}(\Delta s) + \sum_{k=1}^{n}\frac{v_k^{(0)}(\Delta s)}{(k+1)\ell_{k-1}} + s_0, \tag{3}$$

$\{v_n^{(0)}(\Delta s)\}$ is said to be a logarithmic generator sequence of $\{s_n\}$.

For every nonnegative integer $r$, we introduce $t_n^{(r)}(s)$ and $v_n^{(r)}(\Delta s)$ by $t_n^{(r)}(s) = \begin{cases} \dfrac{1}{\ell_n}\displaystyle\sum_{k=0}^{n}\frac{t_k^{(r-1)}(s)}{k+1} & , r \geq 1 \\ s_n & , r = 0 \end{cases}$ and $v_n^{(r)}(\Delta s) = \begin{cases} \dfrac{1}{\ell_n}\displaystyle\sum_{k=0}^{n}\frac{v_k^{(r-1)}(\Delta s)}{k+1} & , r \geq 1 \\ v_n^{(0)}(\Delta s) & , r = 0 \end{cases}$ respectively.

The classical logarithmic control modulo of the oscillatory behavior of $\{s_n\}$ is given by

$$\omega_n^{(0)}(s) = \alpha_n \Delta s_n \sim n \log n \Delta s_n, \tag{4}$$

where $\alpha_n = (n+1)\ell_{n-1}$. The general logarithmic control modulo of the oscillatory behavior of $\{s_n\}$ of integer order $r \geq 1$ is recursively defined by

$$\omega_n^{(r)}(s) = \omega_n^{(r-1)}(s) - t_n^{(1)}(\omega^{(r-1)}(s)). \tag{5}$$

For every nonnegative integer $r$, we have $\left(\alpha_n\Delta\right)_r s_n = \left(\alpha_n\Delta\right)_{r-1}\left(\alpha_n\Delta s_n\right) = \alpha_n\Delta\left(\left(\alpha_n\Delta\right)_{r-1}s_n\right)$, where $\left(\alpha_n\Delta\right)_0 s_n = s_n$ and $\left(\alpha_n\Delta\right)_1 s_n = \alpha_n\Delta s_n$.

**Lemma 1** (Sezer and Çanak, [7]) For every integer $r \geq 1$, the assertion

$$\omega_n^{(r)}(s) = \left(\alpha_n\Delta\right)_r v_n^{(r-1)}(\Delta s)$$

is valid.

**Definition 2** A sequence $\{s_n\}$ is called slowly oscillating with respect to summability $(\ell, 1)$ if

$$\lim_{\lambda\to 1^+}\limsup_{n\to\infty}\max_{n<k\leq[n^\lambda]}|s_k - s_n| = 0 \tag{6}$$

or equivalently

$$\lim_{\lambda\to 1^-}\limsup_{n\to\infty}\max_{[n^\lambda]\leq k<n}|s_n - s_k| = 0, \tag{7}$$

where $[n^\lambda]$ denotes the integer part of $n^\lambda$.

Note that if the two-sided condition $n\log n\Delta s_n = O(1)$ is satisfied, then (6) holds.

There are subsequentially convergent sequences which are not slowly oscillating with respect to summability $(\ell, 1)$, and vice versa. For instance, $\{\log(\log n\})$ is subsequentially convergent but not slowly oscillating with respect to summability $(\ell, 1)$, conversely, the sequence $\left\{\sin\left(\sum_{k=1}^{n} \frac{\log k}{k}\right)\right\}$ is slowly oscillating with respect to summability $(\ell, 1)$ but not subsequentially convergent.

The following lemma indicates that slow oscillation of $\{s_n\}$ is a Tauberian condition for $(\ell, 1)$ summability.

**Lemma 2** If $\{s_n\}$ is $(\ell, 1)$ summable to $L$ and slowly oscillating with respect to summability $(\ell, 1)$, then it converges to the same value.

## MAIN RESULTS

**Theorem 2** If $\{s_n\}$ is bounded and $\{\Delta s_n\}$ is slowly oscillating with respect to summability $(\ell, 1)$, then $\{s_n\}$ is subsequentially convergent.

**Remark 1** Notice that the following conditions are some of the classical Tauberian conditions for the $(\ell, 1)$ summability which imply slow oscillation of $\{\Delta s_n\}$:

(i) $\{s_n\}$ is slowly oscillating with respect to summability $(\ell, 1)$, (Kwee, [8])
(ii) $\omega_n^{(0)}(s) = O(1)$, (Kwee, [8])
(iii) $\omega_n^{(0)}(s) = o(1)$, (Ishiguro, [9])
(iv) $\{v_n^{(0)}(\Delta s)\}$ is slowly oscillating with respect to summability $(\ell, 1)$, (Sezer and Çanak, [7])
(v) $v_n^{(0)}(\Delta s) = o(1)$ (Kwee, [10])

**Theorem 3** If $\{s_n\}$ is bounded and $\{\Delta(t_n^{(1)}(\omega^{(r)}(s)))\}$ is slowly oscillating with respect to summability $(\ell, 1)$ for some nonnegative integer $r$, then $\{s_n\}$ is subsequentially convergent.

**Remark 2** The following results are noteworthy.

(i) If $\{t_n^{(1)}(\omega^{(r)}(s))\}$ is slowly oscillating with respect to summability $(\ell, 1)$, then so is $\{\Delta(t_n^{(1)}(\omega^{(r)}(s)))\}$ of its backward difference.
(ii) Set $r = 0$ in $\{t_n^{(1)}(\omega^{(r)}(s))\}$. Then slow oscillation of $\{v_n^{(0)}(\Delta s)\}= \{t_n^{(1)}(\omega^{(0)}(s))\}$ is sufficient for subsequential convergence of a bounded sequence.
(iii) Two-sided condition $n \log n \Delta v_n^{(0)}(\Delta s) = O(1)$ implies slow oscillation of $\{v_n^{(0)}(\Delta s)\}$.

**Theorem 4** Let $\{s_n\}$ be a bounded sequence and $\{A_n\}$ be a sequence satisfying

$$\frac{1}{\ell_n} \sum_{k=0}^{n} \frac{|A_k|^p}{k+1} = O(1), p > 1. \tag{8}$$

If

$$\omega_n^{(r)}(s) = O(A_n) \tag{9}$$

for some nonnegative integer $r$, then $\{s_n\}$ is subsequentially convergent.

**Remark 3** Considering special cases of Theorem 4, we obtain the following corollaries.

(i) Take $A_n = 1$ for all integer $n \geq 0$. Then (8) and (9) reduce to $\omega_n^{(r)}(s) = O(1)$.
(ii) Take $A_n = \alpha_n \Delta t_n^{(1)}(\omega^{(r)}(s))$, then by the condition

$$\frac{1}{\ell_n} \sum_{k=0}^{n} \frac{|\alpha_k \Delta t_k(\omega^{(r)}(s))|^p}{k+1} = O(1), \ \ p > 1, \tag{10}$$

we get subsequential convergence of a bounded sequence $\{s_n\}$ using Remark 2, since (10) necessitate that $\{t_n^{(1)}(\omega^{(r)}(s))\}$ is slowly oscillating with respect to summability $(\ell, 1)$.

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