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Multidimensional Asymptotically Lacunary Statistical Equivalent of Order α for Sequences of Fuzzy Numbers

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ABSTRACT

The main goal of this article is to present the notion of double asymptotically lacunary statistical equivalent of order α for sequences of fuzzy numbers by considering fuzzy numbers and Pringsheim limit. To accomplish this goal, we mainly investigate some fundamental properties of the newly introduced notion. Additionally, it should be note that some interesting inclusion theorems are examined and also new variations are presented.

KEYWORDS

Asymptotically Statistical Equivalent; Double Lacunary Sequences; Statistical Limit Points; Fuzzy Numbers

2000 MATHEMATICS SUBJECT CLASSIFICATIONS

Primary 40A99; Secondary 40A05

1. Introduction

Fridy [1] put forward the idea of statistical convergence, which is a significant part of the Summability Theory. In 1980, Pobyvancts [2] presented the idea of asymptotically regular matrices. In 2003, Patterson [3] introduced the notions of asymptotically statistically equivalent sequences by combining the notion of asymptotically equivalent introduced by Marouf [4] and statistical convergence.

On the other hand, the notion of convergence for double sequences was presented by Pringsheim in [5].

Definition 1.1: A double sequence $y = (y_{r,s})$ has Pringsheim limit $\varpi \in \mathbb{R}$ (symbolized by $P - \lim_{r,s\to\infty} y_{r,s} = \varpi$) if given $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that $|y_{r,s} - \varpi| < \varepsilon$, whenever $r, s > M$. We shall show such an y shortly as '*P−convergent*'.

Recently, Mursaleen and Edely [6] extended Pringsheim's definition of statistical convergence for double sequences, and Patterson [7] also defined double asymptotically equivalent as follows:

Definition 1.2: Two nonnegative double sequences $(y_{r,s})$ and $(z_{r,s})$ are said to be asymptotically equivalent provided that

$$P - \lim_{r,s} \frac{y_{r,s}}{z_{r,s}} = 1.$$

If this condition is met, it is symbolised by $y \overset{P^2}{\sim} z$.

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Later on the following definition was given by Savaş and Patterson [8].

Definition 1.3: The double sequence $\Phi_{\xi,\eta} = \{(r_\xi, s_\eta)\}$ is named double lacunary provided that there exist two increasing of integers in such a way that $r_0 = 0$, $\gamma_\xi = r_\xi - r_{\xi-1} \to \infty$ whenever $\xi \to \infty$ and $s_0 = 0$, $\overline{\gamma}_\eta = s_\eta - s_{\eta-1} \to \infty$ as $\eta \to \infty$. Also, $r_{\xi,\eta} = r_\xi s_\eta$, $\gamma_{\xi,\eta} = \gamma_\xi \overline{\gamma}_\eta$, $\Phi_{\xi,\eta}$ is determined by

$$J_{\xi,\eta} = \left\{(r,s) : r_{\xi-1} < r \leq r_\xi \& s_{\eta-1} < s \leq s_\eta\right\},$$

$\zeta_\xi = \frac{r_\xi}{r_{\xi-1}}$, $\overline{\zeta}_\eta = \frac{s_\eta}{s_{\eta-1}}$, and $\zeta_{\xi,\eta} = \zeta_\xi \overline{\zeta}_\eta$.

Also, in Ref. [9,10], a different direction was presented to the concept of statistical convergence for double sequences in which the notion of statistical convergence of order α, $0 < \alpha \leq 1$ was introduced by shifting (mn) by $(mn)^\alpha$ in the denominator in the definition of statistical convergence. Please note that the notion of metric space can define as an arbitrary fuzzy set that a distance between all elements of the set are described. It is possible to define many different metrics on the space of fuzzy numbers; nonetheless, the most preferential metric among these metrics is the Hausdorff distance for fuzzy numbers depended on the classical Hausdorff distance between compact convex subsets of R^p.

Recently, fuzzy numbers were studied and generalised more general concepts via statistical convergence in Refs. [11–16].

We now consider $C(R^p) = \{K \subset R^p : K \text{ compact and convex}\}$. The spaces $C(R^p)$ has a linear structure induced by the operations

$$K + H = \{k+h, k \in K, h \in H\}$$

and

$$\varsigma A = \{\varsigma a, \varsigma \in E\}$$

for $K, H \in C(R^p)$ and $\varsigma \in R$. The Hausdorff distance between K and H of $C(R^p)$ is defined as

$$\delta_\infty(K,H) = \max\left\{ \sup_{k\in E} \inf_{h\in H} \|k-h\|,\ \sup_{k\in K} \inf_{h\in H} \|k-h\| \right\}$$

It is obvious that $(C(R^p), \delta_\infty)$ is a complete (not separable) metric space. A fuzzy number is a function Y from R^p to [0, 1] satisfying

(1) Y is normal, in other words, there exists an $\tau_0 \in R^p$ such that $Y(\tau_0) = 1$;
(2) Y is fuzzy convex, in other words, for any $\tau, \upsilon \in R^p$ and $0 \leq \varsigma \leq 1$,

$$Y(\varsigma\tau + (1-\varsigma)\upsilon) \geq \min\{Y(\tau), Y(\upsilon)\};$$

(3) Y is upper semi-continuous;
(4) The closure of $\{\upsilon \in R^p : Y(\tau) > 0\}$; denoted by Y^0, is compact.

These features imply that for each $0 < \beta \leq 1$, the $\beta-$level set

$$Y^\beta = \left\{\tau \in R^p : Y(\tau) > \beta\right\}$$

is a nonempty compact convex, subset of R^p, as is the support Y^0. Let $L(R^p)$ denote the set of all fuzzy numbers. The linear structure of $L(R^p)$ shows addition $Y+Z$ and scalar multiplication ςY, $\varsigma \in R$, in the sense of $\beta-$level sets, by

$$[Y{+}Z]^{\beta} = [Y]^{\beta} + [Z]^{\beta}$$

and

$$[\varsigma Y]^{\beta} = \varsigma\, [Y]^{\beta}$$

for each $0 \leq \beta \leq 1$.

Define for each $1 \leq q < \infty$

$$d_q\,(Y,Z) = \left\{ \int_0^1 \delta_\infty \left(Y^\beta, Z^\beta\right)^q \, \mathrm{d}\beta \right\}^{\frac{1}{q}}$$

and $d_\infty = sup_{0\leq\beta\leq1}\, \delta_\infty(Y^\beta, Z^\beta)$. Clearly $d_\infty(Y,Z) = lim_{q\to\infty}\, d_q(Y,Z)$ with $d_q \leq d_r$ if $q \leq r$. Moreover, d_q is a complete, separable and locally compact metric space [11].

During the paper, d will denote d^q with $1 \leq q \leq \infty$. Also, a metric d on $L(\mathbb{R})$ is said to be a translation invariant if $d(Y+W, Z+W) = d(Y,Z)$ for $Y,Z,W \in L(\mathbb{R})$.

Provided that d is a translation invariant metric on $L(\mathbb{R})$, then it is clear to see that $d(Y + Z, 0) \leq d(Y,0) + d(Z,0)$.

2. Main Results

In this section, we will present new concepts and examine the relationship among these concepts.

Definition 2.1: Let $\Phi_{\xi,\eta} = \{(r_\xi, s_\eta)\}$ be a double lacunary sequence; and $p = (p_{u,v})$ be a sequence of positive real numbers; two sequences Y and Z of fuzzy numbers are double strongly asymptotically equivalent of order α to multiple ϖ, where $0 < \alpha \leq 1$ provided that

$$P - \lim_{\xi,\eta\to\infty} \frac{1}{\gamma^{\alpha}_{\xi,\eta}} \sum_{(u,v)\in J_{\xi,\eta}} d\left(\frac{Y_{u,v}}{Z_{u,v}}, \varpi\right)^{p_{u,v}} = 0$$

(denoted by $Y \overset{N_\Phi^{\varpi^\alpha}(p)}{\sim} Z$), and simply double lacunary strongly asymptotically equivalent of order α if $\varpi = 1$. If we take $p_{u,v} = p$ for all u and v, then we write $Y \overset{N_\Phi^{\varpi^\alpha_p}}{\sim} Z$ instead of $Y \overset{N_\Phi^{\varpi^\alpha}(p)}{\sim} Z$.

Definition 2.2: Let $p = (p_{u,v})$ be a sequence of positive real numbers, and we consider two sequences Y and Z of fuzzy numbers. The two sequences Y and Z are called as double strongly Cesáro summable of order α to ϖ, where $0 < \alpha \leq 1$ provided that

$$P - \lim_{m,n} \frac{1}{(mn)^{\alpha}} \sum_{u,v=1,1}^{m,n} d\left(\frac{Y_{u,v}}{Z_{u,v}}, \varpi\right)^{p_{u,v}} = 0$$

(denoted by $Y \overset{\sigma^\alpha(p)}{\sim} Z$) and in simple terms double strongly Cesáro asymptotically equivalent of order α if $\varpi = 1$.

Definition 2.3: Let $\Phi_{\xi,\eta} = \{(r_\xi, s_\eta)\}$ be a double lacunary sequence; the two nonnegative sequences Y and Z are called as double asymptotically lacunary statistical equivalent of order α to ϖ, where $0 < \alpha \leq 1$ if for every $\epsilon > 0$,

$$P - \lim_{\xi,\eta\to\infty} \frac{1}{\gamma^{\alpha}_{\xi,\eta}} \left|\left\{(u,v) \in J_{\xi,\eta} : d\left(\frac{Y_{u,v}}{Z_{u,v}}, \varpi\right) \geq \epsilon\right\}\right| = 0$$

(denoted by $Y \overset{S^{\varpi^{\alpha}}_{\Phi}}{\sim} Z$) and simply asymptotically lacunary statistical equivalent, if $\varpi = 1$.

Let us prove the following theorems.

Theorem 2.1: *Let $\Phi_{\xi,\eta}$ be a double lacunary sequence. Then*

(1) *If $Y \overset{N^{\varpi^{\alpha}_{p}}_{\Phi}}{\sim} Z$ then $Y \overset{S^{\varpi^{\alpha}}_{\Phi}}{\sim} Z$.*

(2) *If $Y, Z \in l^2_\infty(F^*)$ and $Y \overset{S^{\varpi^{\alpha}}_{\Phi}}{\sim} Z$ then $Y \overset{N^{\varpi^{\alpha}_{p}}_{\Phi}}{\sim} Z$.*

Proof: (1) If $\epsilon > 0$ and $Y \overset{N^{\varpi^{\alpha}_{p}}_{\Phi}}{\sim} Z$ then

$$\frac{1}{\gamma^{\alpha}_{\xi,\eta}} \sum_{(u,v)\in J_{\xi,\eta}} d\left(\frac{Y_{u,v}}{Z_{u,v}}, \varpi\right)^{p_{u,v}} \geq \frac{1}{\gamma^{\alpha}_{\xi,\eta}} \sum_{(u,v)\in J_{\xi,\eta} \& d\left(\frac{Y_{u,v}}{Z_{u,v}} - \varpi\right)\geq\epsilon} d\left(\frac{Y_{u,v}}{Z_{u,v}}, \varpi\right)^{p_{u,v}}$$

$$\geq \epsilon^p \frac{1}{\gamma^{\alpha}_{\xi,\eta}} \left|\left\{(u,v) \in J_{\xi,\eta} : d\left(\frac{Y_{u,v}}{Z_{u,v}}, \varpi\right) \geq \epsilon\right\}\right|.$$

Therefore, $Y \overset{S^{\varpi^{\alpha}}_{\Phi}}{\sim} Z$. (2) Suppose that Y and Z are in $l^2_\infty(F)$ and $Y \overset{S^{\varpi^{\alpha}}_{\Phi}}{\sim} Z$, then we can presume that $d(\frac{Y_{u,v}}{Z_{u,v}}, \varpi) \leq M$ for u and v. Let $\epsilon > 0$ be given and N_ϵ be such that

$$\frac{1}{\gamma^{\alpha}_{\xi,\eta}} \left|\left\{(u,v) \in J_{\xi,\eta} : d\left(\frac{Y_{u,v}}{Z_{u,v}}, \varpi\right) \geq \left(\frac{\epsilon}{2}\right)^{\frac{1}{p}}\right\}\right| \leq \frac{\epsilon}{2M^p}$$

for all $\xi, \eta > N_\epsilon$ and let

$$L_{u,v} := \left\{(u,v) \in J_{\xi,\eta} : d\left(\frac{Y_{u,v}}{Z_{u,v}}, \varpi\right) \geq \left(\frac{\epsilon}{2}\right)^{\frac{1}{p}}\right\}.$$

Now for all $\xi, \eta > N$ we are granted

$$\frac{1}{\gamma^{\alpha}_{\xi,\eta}} \sum_{(u,v)\in J_{\xi,\eta}} d\left(\frac{Y_{u,v}}{Z_{u,v}} - \varpi\right)^{p_{u,v}} = \frac{1}{\gamma^{\alpha}_{\xi,\eta}} \sum_{u,v\in L_{u,v}} d\left(\frac{Y_{u,v}}{Z_{u,v}}, \varpi\right)^{p_{u,v}}$$

$$+ \frac{1}{\gamma^{\alpha}_{\xi,\eta}} \sum_{u,v\notin L_{u,v}} d\left(\frac{Y_{u,v}}{Z_{u,v}}, \varpi\right)^{p_{u,v}}$$

$$\geq \frac{1}{\gamma^{\alpha}_{\xi,\eta}} \frac{\gamma^{\alpha}_{\xi,\eta}\epsilon}{2M^p} M^p + \frac{1}{\gamma^{\alpha}_{\xi,\eta}} \gamma^{\alpha}_{\xi,\eta} \frac{\epsilon}{2}.$$

Hence, we obtain $Y \overset{N^{\varpi^{\alpha}_{p}}_{\Phi}}{\sim} Z$. ∎

Theorem 2.2: *Let $\Phi_{\xi,\eta}$ be a double lacunary sequence and $\sup_{u,v} p_{u,v} = H$ then $Y \overset{N_{\Phi}^{\varpi_p^\alpha}}{\sim} Z$ implies $Y \overset{S_{\Phi}^{\varpi^\alpha}}{\sim} Z$.*

Proof: Let $Y \overset{N_{\Phi}^{\varpi_p^\alpha}}{\sim} Z$ and $\epsilon > 0$ be given. Then

$$
\begin{aligned}
\frac{1}{\gamma_{\xi,\eta}^{\alpha}} \sum_{(u,v)\in J_{\xi,\eta}} d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)^{p_{u,v}} &= \frac{1}{\gamma_{\xi,\eta}^{\alpha}} \sum_{(u,v)\in J_{\xi,\eta} \& d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)\geq\epsilon} d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)^{p_{u,v}} \\
&\quad + \frac{1}{\gamma_{\xi,\eta}^{\alpha}} \sum_{(u,v)\in J_{\xi,\eta} \& d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)<\epsilon} d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)^{p_{u,v}} \\
&\geq \frac{1}{\gamma_{\xi,\eta}^{\alpha}} \sum_{(u,v)\in J_{\xi,\eta} \& d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)\geq\epsilon} d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)^{p_{u,v}} \\
&\geq \frac{1}{\gamma_{\xi,\eta}^{\alpha}} \sum_{(u,v)\in J_{\xi,\eta} \& d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)\geq\epsilon} (\epsilon)^{p_{u,v}} \\
&\geq \frac{1}{\gamma_{\xi,\eta}^{\alpha}} \sum_{(u,v)\in J_{\xi,\eta} \& d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)\geq\epsilon} \min\{(\epsilon)^{\inf p_{u,v}}, (\epsilon)^{H}\} \\
&\geq \frac{1}{\gamma_{\xi,\eta}^{\alpha}} \left|\left\{(u,v) \in J_{\xi,\eta} : d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right) \geq \epsilon\right\}\right| \min\left\{(\epsilon)^{\inf p_{u,v}}, (\epsilon)^{H}\right\}.
\end{aligned}
$$

Hence, $Y \overset{S_{\Phi}^{\varpi^\alpha}}{\sim} Z$ ■

Theorem 2.3: *Let Y and Z be bounded and $0 < h = \inf_{u,v} p_{u,v} \leq \sup_{u,v} p_{u,v} = H < \infty$. Then $Y \overset{S_{\Phi}^{\varpi^\alpha}}{\sim} Z$ implies $Y \overset{N_{\Phi}^{\varpi_p^\alpha}}{\sim} Z$.*

Proof: Suppose that Y and Z be bounded and $\epsilon > 0$. Since Y and Z are bounded, there is an integer $\widetilde{K}$ such that $d(\frac{Y_{u,v}}{Z_{u,v}},\varpi) \leq \widetilde{K}$ for all u and v; then

$$
\begin{aligned}
\frac{1}{\gamma_{\xi,\eta}^{\alpha}} \sum_{(u,v)\in J_{\xi,\eta}} d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)^{p_{u,v}} &= \frac{1}{\gamma_{\xi,\eta}^{\alpha}} \sum_{(u,v)\in J_{\xi,\eta} \& d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)\geq\epsilon}^{p_{u,v}} d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)^{p_{u,v}} \\
&\quad + \frac{1}{\gamma_{\xi,\eta}^{\alpha}} \sum_{(u,v)\in J_{\xi,\eta} \& d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)<\epsilon}^{p_{u,v}} d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)^{p_{u,v}} \\
&\leq \frac{1}{\gamma_{\xi,\eta}^{\alpha}} \sum_{(u,v)\in J_{\xi,\eta} \& d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)\geq\epsilon} \max\{\widetilde{K}^{h}, \widetilde{K}^{H}\}
\end{aligned}
$$

$$+\frac{1}{\gamma^{\alpha}_{\xi,\eta}}\sum_{(u,v)\in J_{\xi,\eta}\&d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)<\epsilon}\max\{\epsilon\}^{p_{u,v}}$$

$$\leq \max\{\widetilde{K}^{h},\widetilde{K}^{H}\}\frac{1}{\gamma^{\alpha}_{\xi,\eta}}\left|\left\{(u,v)\in J_{\xi,\eta}:d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)\geq\epsilon\right\}\right|$$

$$+\max\{\epsilon^{h},\epsilon^{H}\}.$$

Hence, $Y \overset{N_{\Phi}^{\varpi_p^{\alpha}}}{\sim} Z$. ■

Theorem 2.4: *Let $\Phi_{\xi,\eta}$ be a double lacunary sequence with* $\liminf_{\xi}\zeta_{\xi}>1$ *and* $\liminf_{\eta}\overline{\zeta}_{\eta}>1$, *then $Y \overset{\sigma^{\alpha}(p)}{\sim} Z$ implies $Y \overset{N_{\Phi}^{\varpi_p^{\alpha}}}{\sim} Z$.*

Proof: Suppose $\liminf_{\xi}\zeta_{\xi}>1$ and $\liminf_{\eta}\overline{\zeta}_{\eta}>1$; then there is a $\delta>0$ such that $\zeta_{\xi}>1+\delta$ and $\overline{\zeta}_{\eta}>1+\delta$. This applies $\frac{\gamma_{\xi}}{r_{\xi}}\geq\frac{\delta}{1+\delta}$ and $\frac{\overline{\gamma}_{\eta}}{s_{\eta}}\geq\frac{\delta}{1+\delta}$. Then for $Y \overset{\sigma^{\alpha}(p)}{\sim} Z$,

$$A_{\xi,\eta}=\frac{1}{\gamma^{\alpha}_{\xi,\eta}}\sum_{(\xi,\eta)\in J_{\xi,\eta}}d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)^{p_{u,v}}$$

$$=\frac{1}{\gamma^{\alpha}_{\xi,\eta}}\sum_{r=1}^{r_{\xi}}\sum_{s=1}^{s_{\eta}}d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)^{p_{u,v}}$$

$$-\frac{1}{\gamma^{\alpha}_{\xi,\eta}}\sum_{r=1}^{r_{\xi-1}}\sum_{s=1}^{s_{\eta-1}}d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)^{p_{u,v}}$$

$$-\frac{1}{\gamma^{\alpha}_{\xi,\eta}}\sum_{r=r_{\xi-1}+1}^{r_{\xi}}\sum_{s=1}^{s_{\eta-1}}d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)^{p_{u,v}}$$

$$-\frac{1}{\gamma^{\alpha}_{\xi,\eta}}\sum_{r=1}^{r_{\xi-1}}\sum_{s=s_{\eta-1}+1}^{s_{\eta}}d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)^{p_{u,v}}$$

$$=\frac{\left(r_{\xi}s_{\eta}\right)^{\alpha}}{\gamma^{\alpha}_{\xi,\eta}}\frac{1}{\left(r_{\xi}s_{\eta}\right)^{\alpha}}\sum_{r=1}^{r_{\xi}}\sum_{s=1}^{s_{\eta}}d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)^{p_{u,v}}$$

$$-\frac{\left(r_{\xi-1}s_{\eta-1}\right)^{\alpha}}{\overline{\gamma}^{\alpha}_{\eta}}\frac{1}{\left(r_{\xi-1}s_{\eta-1}\right)^{\alpha}}\sum_{r=1}^{r_{\xi-1}}\sum_{s=1}^{s_{\eta-1}}d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)^{p_{u,v}}$$

$$-\frac{1}{\gamma^{\alpha}_{\xi}}\sum_{r=r_{\xi-1}+1}^{r_{\xi}}\frac{\left(s_{\eta-1}\right)^{\alpha}}{\overline{\gamma}^{\alpha}_{\eta}}\frac{1}{\left(s_{\eta-1}\right)^{\alpha}}\sum_{s=1}^{s_{\eta-1}}d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)^{p_{u,v}}$$

$$-\frac{1}{\overline{\gamma}^{\alpha}_{s}}\sum_{s=s_{\eta-1}+1}^{s_{\eta}}\frac{\left(r_{\xi-1}\right)^{\alpha}}{\gamma^{\alpha}_{\xi}}\frac{1}{\left(r_{\xi-1}\right)^{\alpha}}\sum_{r=1}^{r_{\xi-1}}d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)^{p_{u,v}}$$

Since $Y \overset{N_\Phi^{\varpi_p^\alpha}}{\sim} Z$ the last two terms tends to zero in the Pringsheim sense, thus

$$A_{\xi,\eta} = \frac{\left(r_\xi s_\eta\right)^\alpha}{\gamma_{\xi,\eta}^\alpha} \frac{1}{\left(r_\xi s_\eta\right)^\alpha} \sum_{r=1}^{r_\xi} \sum_{s=1}^{s_\eta} d\left(\frac{Y_{u,v}}{Z_{u,v}}, \varpi\right)^{p_{u,v}} - \frac{\left(r_{\xi-1} s_{\eta-1}\right)^\alpha}{\gamma_{\xi,\eta}^\alpha} \frac{1}{\left(r_{\xi-1} s_{\eta-1}\right) \alpha} \sum_{r=1}^{r_{\xi-1}} \sum_{s=1}^{s_{\eta-1}} d\left(\frac{Y_{u,v}}{Z_{u,v}}, \varpi\right)^{p_{u,v}} + o(1).$$

Since $\gamma_{\xi,\eta}^\alpha = (r_\xi s_\eta)^\alpha - (r_{\xi-1} s_{\eta-1})^\alpha$ we are granted the following:

$$\frac{\left(r_\xi s_\eta\right)^\alpha}{\gamma_{\xi,\eta}^\alpha} \leq \frac{1+\delta}{\delta} \quad \text{and} \quad \frac{\left(r_{\xi-1} s_{\eta-1}\right)^\alpha}{\gamma_{\xi,\eta}^\alpha} \leq \frac{1}{\delta}.$$

The terms

$$\frac{1}{\left(r_\xi s_\eta\right)^\alpha} \sum_{r=1}^{r_\xi} \sum_{s=1}^{s_\eta} d\left(\frac{Y_{u,v}}{Z_{u,v}}, \varpi\right)^{p_{u,v}}$$

and

$$\frac{1}{\left(r_{\xi-1} s_{\eta-1}\right)^\alpha} \sum_{r=1}^{r_{\xi-1}} \sum_{s=1}^{s_{\eta-1}} d\left(\frac{Y_{u,v}}{Z_{u,v}}, \varpi\right)^{p_{u,v}}$$

are both Pringsheim null sequences. Hence, $A_{\xi,\eta}$ is a Pringsheim null sequence. Consequently, $Y \overset{N_\Phi^{\varpi_p^\alpha}}{\sim} Z$. ■

Theorem 2.5: *Let $\Phi_{\xi,\eta}$ be a double lacunary sequence with $\limsup_\xi \zeta_\xi < \infty$ and $\limsup_\eta \overline{\zeta}_\eta < \infty$, then $Y \overset{N_\Phi^{\varpi_p^\alpha}}{\sim} Z$ implies $Y \overset{\sigma^\alpha(p)}{\sim} Z$.*

Proof: Since $\limsup_\xi \frac{r_\xi}{r_{\xi-1}} < \infty$ and $\limsup_\eta \frac{s_\eta}{s_{\eta-1}} < \infty$ there exists $\widetilde{H} > 0$ such that $\frac{r_\xi}{r_{\xi-1}} < \widetilde{H}$ and $\frac{s_\eta}{s_{\eta-1}} < \widetilde{H}$ for all ξ and η. Let $Y \overset{N_\Phi^{\varpi_p^\alpha}}{\sim} Z$ and $\epsilon > 0$ and there exist $\xi_0 > 0$ and $\eta_0 > 0$ such that for every $i \geq \xi_0$ and $j \geq \eta_0$

$$A_{i,j} = \frac{1}{\gamma_{\xi,\eta}^\alpha} \sum_{(u,v)\in J_{\xi,\eta}} d\left(\frac{Y_{u,v}}{Z_{u,v}}, \varpi\right)^{p_{u,v}} < \epsilon.$$

Let $T = \max\{A_{i,j} : 1 \leq \xi \leq \xi_0 \text{ and } 1 \leq \eta \leq \eta_0\}$, and q and w be such that $r_{\xi-1} < q \leq r_\xi$ and $s_{\eta-1} < w \leq s_\eta$. Thus,

$$\begin{aligned}
&\frac{1}{(qw)^\alpha}\sum_{r,s=1,1}^{m,n} d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)^{p_{u,v}} \\
&\leq \frac{1}{\left(r_{\xi-1}s_{\eta-1}\right)^\alpha}\sum_{r,s=1,1}^{r_\xi s_\eta} d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)^{p_{u,v}} \\
&\leq \frac{1}{\left(r_{\xi-1}s_{\eta-1}\right)^\alpha}\sum_{t,w=1,1}^{\xi,\eta}\left(\sum_{(u,v)\in J_{t,w}} d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)^{p_{u,v}}\right) \\
&= \frac{1}{\left(r_{\xi-1}s_{\eta-1}\right)^\alpha}\sum_{t,w=1,1}^{\xi_0,\eta_0}\gamma_{t,w}A_{t,u} + \frac{1}{\left(r_{\xi-1}s_{\eta-1}\right)^\alpha}\sum_{(\xi_0<t\leq\xi)\cup(\eta_0<w\leq\eta)}\gamma_{t,w}A_{t,w} \\
&\leq \frac{T}{\left(r_{\xi-1}s_{\eta-1}\right)^\alpha}\sum_{t,w=1,1}^{\xi_0,\eta_0}\gamma_{t,u} + \frac{1}{\left(r_{\xi-1}s_{\eta-1}\right)^\alpha}\sum_{(\xi_0<t\leq\xi)\cup(\eta_0<w\leq\eta)}\gamma_{t,w}A_{t,w} \\
&\leq \frac{Tr_{\xi_0}s_{\eta_0}\xi_0\eta_0}{\left(r_{\xi-1}s_{\eta-1}\right)^\alpha} + \frac{1}{\left(r_{\xi-1}s_{\eta-1}\right)^\alpha}\sum_{(\xi_0<t\leq\xi)\cup(\eta_0<w\leq\eta)}A_{t,w}\gamma_{t,w} \\
&\leq \frac{Tr_{\xi_0}s_{\eta_0}\xi_0\eta_0}{\left(r_{\xi-1}s_{\eta-1}\right)^\alpha} + \left(\sup_{t\geq\xi_0\cup w\geq\eta_0} A_{t,w}\right)\frac{1}{\left(r_{\xi-1}s_{\eta-1}\right)^\alpha}\sum_{(\xi_0<t\leq\xi)\cup(\eta_0<w\leq\eta)}\gamma_{t,w} \\
&\leq \frac{Tr_{\xi_0}s_{\eta_0}\xi_0\eta_0}{\left(r_{\xi-1}s_{\eta-1}\right)^\alpha} + \epsilon\sum_{(\xi_0<t\leq\xi)\cup(\eta_0<w\leq\eta)}A_{t,w} \\
&\leq \frac{Tr_{\xi_0}s_{\eta_0}\xi_0\eta_0}{\left(r_{\xi-1}s_{\eta-1}\right)^\alpha} + \epsilon\widetilde{H}^2.
\end{aligned}$$

Since r_ξ and s_η both approach infinity as both q and w approach infinity. Therefore

$$\frac{1}{(qw)^\alpha}\sum_{u,v=1,1}^{q,w} d\left(\frac{Y_{u,v}}{Z_{u,v}},\varpi\right)^{p_{u,v}} \to 0.$$

Therefore, $Y \overset{\sigma^\alpha(p)}{\sim} Z$. ∎

Disclosure statement

The Author declares that there is no conflict of interest.

Notes on contributor

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