



Axial-vector and pseudoscalar tetraquarks $[ud][\bar{c}\bar{s}]$

H. Sundu^{1,2}, S. S. Agaev³, K. Azizi^{4,5,a}

¹ Department of Physics, Kocaeli University, 41380 Izmit, Turkey

² Department of Physics Engineering, Istanbul Medeniyet University, 34700 Istanbul, Turkey

³ Institute for Physical Problems, Baku State University, 1148 Baku, Azerbaijan

⁴ Department of Physics, University of Tehran, North Karegar Avenue, Tehran 14395-547, Iran

⁵ Department of Physics, Doğuş University, Acibadem-Kadıköy, 34722 Istanbul, Turkey

Received: 21 September 2022 / Accepted: 18 February 2023 / Published online: 6 March 2023
© The Author(s) 2023

Abstract Spectroscopic parameters and widths of the fully open-flavor axial-vector and pseudoscalar tetraquarks X_{AV} and X_{PS} with content $[ud][\bar{c}\bar{s}]$ are calculated by means of the QCD sum rule methods. Masses and current couplings of X_{AV} and X_{PS} are found using two-point sum rule computations performed by taking into account various vacuum condensates up to dimension 10. The full width of the axial-vector state X_{AV} is evaluated by including into analysis S -wave decay modes $X_{AV} \rightarrow D^*(2010)^- K^+$, $\bar{D}^*(2007)^0 K^0$, $D^- K^*(892)^+$, and $\bar{D}^0 K^*(892)^0$. In the case of X_{PS} , we consider S -wave decay $X_{PS} \rightarrow \bar{D}_0^*(2300)^0 K^0$, and P -wave processes $X_{PS} \rightarrow D^- K^*(892)^+$ and $X_{PS} \rightarrow \bar{D}^0 K^*(892)^0$. To determine partial widths of these decay modes, we employ the QCD light-cone sum rule method and soft-meson approximation, which are necessary to estimate strong couplings at tetraquark–meson–meson vertices $X_{AV} D^- D^*(2010)^- K^+$, etc. Our predictions for the mass $m_{AV} = (2800 \pm 75)$ MeV and width $\Gamma_{AV} = (58 \pm 10)$ MeV of the tetraquark X_{AV} , as well as results $m_{PS} = (3000 \pm 60)$ MeV and $\Gamma_{PS} = (65 \pm 12)$ MeV for the same parameters of X_{PS} may be useful in future experimental studies of multi-quark hadrons.

1 Introduction

Recent LHCb information on new structures $X_0(2900)$ and $X_1(2900)$ observed in the invariant $D^- K^+$ mass distribution of the process $B^+ \rightarrow D^+ D^- K^+$ [1,2], enhanced activity of researches to investigate fully open-flavor exotic mesons. In fact, by taking into account dominant decays of the resonance-like peaks $X_{0(1)}(2900) \rightarrow D^- K^+$ and assuming that they are four-quark systems, one sees that $X_{0(1)}(2900)$ are built of quarks c , s , u , and d . The LHCb

collaboration measured masses and widths of the structures $X_{0(1)}(2900)$, and fixed their spin-parities. It was found that $X_0(2900)$ and $X_1(2900)$ bear the quantum numbers $J^P = 0^+$ and $J^P = 1^-$, respectively. Let us note that, alternatively, structures $X_{0(1)}(2900)$ may appear as triangle singularities in some rescattering diagrams: such interpretation was not excluded by LHCb as well.

In the four-quark picture, widely accepted to explain the LHCb data, $X_{0(1)}(2900)$ may be considered in the framework of both the molecule and diquark–antidiquark (tetraquark) models. Thus, in publications [3,4] the resonance $X_0(2900)$ was analyzed as the scalar diquark–antidiquark state $[ud][\bar{c}\bar{s}]$, whereas in Refs. [5,6] it was treated as a molecule $D^{*-} K^{*+}$ or $\bar{D}^{*0} K^{*0}$. The situation is almost the same for the vector resonance $X_1(2900)$: it was studied in the context of the tetraquark and molecule models, for instance, in Refs. [5,7,8]. There are numerous articles devoted to investigations of $X_{0(1)}(2900)$ using different methods and schemes of high energy physics: relatively complete list of such papers can be found in Refs. [6,7,9].

An interesting conjecture about nature of $X_0(2900)$ was made in Ref. [10], where it was interpreted as a radial excitation X'_0 of the scalar tetraquark $X_0 = [ud][\bar{c}\bar{s}]$. We addressed this problem in our work [9], and calculated masses and widths of the ground-state $1S$ and radially excited $2S$ tetraquarks $X_0^{(\prime)}$ using the QCD sum rule method. We modeled $X_0^{(\prime)}$ as particles composed of the axial-vector diquark $[ud]$ and axial-vector antidiquark $[\bar{c}\bar{s}]$. We also constructed the tetraquarks $X_S^{(\prime)}$ by utilizing a scalar diquark and antidiquark, and found their parameters. It was demonstrated that, the ground-state particles X_0 and X_S are lighter than the resonance $X_0(2900)$, whereas radially excited tetraquarks X'_0 and X'_S with the masses around ≈ 3320 MeV are heavier it.

^a e-mail: kazem.azizi@ut.ac.ir (corresponding author)

In other words, none of these four-quark states can be identified with the resonance $X_0(2900)$. Therefore, it is reasonable to treat X_0 and X_S as new hypothetical exotic mesons to be searched for in experiments.

Fully open-flavor tetraquarks, to be fair, were already objects of theoretical studies, which intensified after information on the resonance $X(5568)$ presumably composed of b , s , u , and d quarks [11]. Though existence of $X(5568)$ was not confirmed by other experimental groups, its charmed partners $b \rightarrow c$ are still under detailed analysis. In fact, the spectroscopic parameters and full width of scalar tetraquark $X_c = [su][\bar{c}\bar{d}]$ were calculated in Ref. [12]. Masses of exotic mesons with the same content, but quantum numbers $J^P = 0^+$ and $J^P = 1^+$ were estimated in Ref. [13].

In various combinations c , s , u , and d quarks form different classes of four-quark mesons, features of which deserve investigations. Interesting class of fully open-flavor particles is collection of states $Z^{++} = [cu][\bar{s}\bar{d}]$, which carries two units of electric charge. The scalar, pseudoscalar, axial-vector and vector members of this group were studied in our articles Refs. [14, 15], respectively. It was pointed out that scalar and vector tetraquarks Z_S^{++} and Z_V^{++} may be observed in the D^+K^+ mass distribution of the decay $B^+ \rightarrow D^-D^+K^+$ [15].

Tetraquarks with a content $[ud][\bar{c}\bar{s}]$ establish new class of open-flavor particles. Observation of the resonances $X_{0(1)}(2900)$ by the LHCb collaboration, available experimental data, and possible interpretation of $X_1(2900)$ as a vector state $X_V = [ud][\bar{c}\bar{s}]$ make these particles objects of special interest. In the present article, we continue our studies started in Refs. [7, 9] by calculating spectroscopic parameters and full widths of axial-vector and pseudoscalar four-quark states X_{AV} and X_{PS} with the same $[ud][\bar{c}\bar{s}]$ content.

Masses and current couplings of X_{AV} and X_{PS} are evaluated in the context of the two-point sum rule method [16, 17]. In calculations, we take into account various vacuum condensates up to dimension 10. The full width of the axial-vector state X_{AV} is found by including into analysis its S -wave decay modes $X_{AV} \rightarrow D^*(2010)^-K^+$, $\bar{D}^*(2007)^0K^0$, $D^-K^*(892)^+$, and $\bar{D}^0K^*(892)^0$. To estimate width of the pseudoscalar tetraquark X_{PS} , we consider kinematically allowed S -wave channel $X_{PS} \rightarrow \bar{D}_0^*(2300)^0K^0$, and P -wave decay modes $X_{PS} \rightarrow D^-K^*(892)^+$ and $X_{PS} \rightarrow D^0K^*(892)^0$.

Partial widths of these processes are governed by strong couplings at relevant vertices, for example, at $X_{AV}D^*(2010)^-K^+$ for the first process. To calculate required couplings, we use the QCD light-cone sum rule (LCSR) method [18] and soft-meson approximation [19, 20]. The latter is necessary to treat tetraquark–meson–meson vertices, which due to unequal number of quark fields in tetraquark and meson interpolating currents differ from standard three-meson vertices [21].

This paper is organized in the following manner: In Sect. 2, we calculate the masses and current couplings of the tetraquarks X_{AV} and X_{PS} . In Sect. 3, we determine strong couplings g_i , $i = 1, 2, 3, 4$ corresponding to vertices $X_{AV}D^*(2010)^-K^+$, $X_{AV}\bar{D}^*(2007)^0K^0$, $X_{AV}D^-K^*(892)^+$, and $X_{AV}\bar{D}^0K^*(892)^0$. In this section, we compute partial widths of corresponding processes, and estimate full width of X_{AV} . In Sect. 4, we consider the decays $X_{PS} \rightarrow \bar{D}_0^*(2300)^0K^0$, $D^-K^*(892)^+$, and $D^0K^*(892)^0$, and find strong couplings G_j , $j = 1, 2, 3$ at relevant vertices. Using G_j , we calculate partial width of these decays and evaluate full width of X_{PS} by saturating it with these channels. Section 5 contains our conclusions.

2 Masses and current couplings of the tetraquarks X_{AV} and X_{PS}

In this section, we compute spectroscopic parameters of the states X_{AV} and X_{PS} by means of the two-point sum rule method. It is an effective nonperturbative approach elaborated to evaluate parameters of ordinary mesons and baryons. The QCD sum rules express various physical quantities in terms of universal vacuum condensates which do not depend on a problem under consideration. At the same time, they contain auxiliary parameters s_0 and M^2 specific for each computation. The first of them is the continuum subtraction parameter s_0 that separates contribution of a ground-state particle in the phenomenological side of a sum rule from effects of higher resonances and continuum states. The Borel parameter M^2 is required to suppress these unwanted continuum effects. By introducing M^2 and s_0 into analysis and employing an assumption about quark-hadron duality one connects phenomenological and QCD sides of sum rules and gets a sum equality. The latter can be used to express physical observables in terms of different vacuum condensates. The parameters M^2 and s_0 generate theoretical uncertainties in results, which nevertheless can be estimated and kept under control.

In what follows, we calculate the mass m and current coupling f of the axial-vector meson X_{AV} (we employ also m_{AV} and f_{AV}), and provide only final results for X_{PS} . The starting point in computation of the spectroscopic parameters of the tetraquark X_{AV} is the correlation function

$$\Pi_{\mu\nu}(p) = i \int d^4x e^{ipx} \langle 0 | \mathcal{T} \{ J_\mu(x) J_\nu^\dagger(0) \} | 0 \rangle. \quad (1)$$

where, $\mathcal{T}$ is the time-ordered product of two currents, and $J_\mu(x)$ is the interpolating current for the axial-vector state X_{AV} . We model the tetraquark X_{AV} as a compound formed by the scalar diquark $u^T C \gamma_5 d$ and axial-vector antidiquark $\bar{c} \gamma_\mu C \bar{s}^T$, which are antitriplet and triplet states of the color group $SU_c(3)$, respectively. Therefore, corresponding interpolating current is given by the formula

$$J_\mu(x) = \epsilon \tilde{u}_b^T(x) C \gamma_5 d_c(x) \bar{c}_d(x) \gamma_\mu C \bar{s}_e^T(x), \quad (2)$$

and belongs to $[\bar{3}_c]_{ud} \otimes [3_c]_{\bar{c}s}$ representation of the color group. In expression above, $\epsilon \tilde{c} = \epsilon_{abc} \epsilon_{ade}$, where a, b, c, d and e are color indices. In Eq. (2) $c(x)$, $s(x)$, $u(x)$ and $d(x)$ denote quark fields, and C is the charge conjugation matrix.

The phenomenological side of the sum rule $\Pi_{\mu\nu}^{\text{Phys}}(p)$

$$\Pi_{\mu\nu}^{\text{Phys}}(p) = \frac{\langle 0 | J_\mu | X_{AV}(p, \varepsilon) \rangle \langle X_{AV}(p, \varepsilon) | J_\nu^\dagger | 0 \rangle}{m^2 - p^2} + \dots, \quad (3)$$

is derived from Eq. (1) by inserting a complete set of intermediate states with quark contents and spin-parity of the tetraquark X_{AV} , and carrying out integration over x . The momentum and polarization vector of X_{AV} are denoted by p and ε , respectively. It should be noted that in $\Pi_{\mu\nu}^{\text{Phys}}(p)$ the ground-state term is written down explicitly, whereas contributions of higher resonances and continuum states are shown by ellipses.

In Eq. (3), we have assumed that the phenomenological side of the sum rule $\Pi_{\mu\nu}^{\text{Phys}}(p)$ can be approximated by a single pole term. But in the case of multiquark systems this approximation has to be used with some caution, because the physical side receives contribution also from two-hadron reducible terms. Indeed, a relevant interpolating current couples not only to a multiquark hadron, but also to a two-hadron continuum. This problem was raised in Refs. [22, 23] when considering pentaquarks, and revisited recently in the case of tetraquarks [24], where it is argued that the contributions at the orders $\mathcal{O}(1)$ and $\mathcal{O}(\alpha_s)$ in the operator product expansion (OPE) are canceled out exactly by the meson–meson scattering states at the hadronic side and the tetraquark molecular states start to receive contributions at the order $\mathcal{O}(\alpha_s^2)$. Then the reducible contributions should be subtracted from the sum rule, which can be done by means of two methods. One of them is direct subtraction of two-hadron terms from $\Pi_{\mu\nu}^{\text{Phys}}(p)$ by calculating current-two-hadron coupling constant using an independent QCD sum rule. This strategy was realized, for example, in Ref. [25] to investigate anti-charmed pentaquark state. Existence of a two-hadron continuum below a multiquark system means that such particle is unstable and decays to these conventional hadrons. In other words, a two-hadron continuum generates the finite width $\Gamma(p^2)$ of a multiquark system. Relevant effects can be taken into account by modifying the quark propagator in Eq. (3)

$$\frac{1}{m^2 - p^2} \rightarrow \frac{1}{m^2 - p^2 - i\sqrt{p^2}\Gamma(p)}. \quad (4)$$

This second method was used to study the tetraquarks [26]. Rather detailed investigations demonstrated that effects of the modification Eq. (4) can be taken into account by absorbing two-meson contributions into a current-tetraquark cou-

pling constant and keeping stable tetraquark's mass [27, 28]. Uncertainties generated by changing of a coupling are numerically smaller than theoretical errors of sum rule analysis itself. In fact, two-meson effects lead to additional $\approx 7\%$ uncertainty in the current coupling f_T for doubly charmed pseudoscalar tetraquark $cc\bar{s}\bar{s}$ with the mass $m_T = 4390$ MeV and full width $\Gamma_T \approx 300$ MeV [27]. In the case of the resonance $Z_c^-(4100)$ these uncertainties do not exceed $\approx 5\%$ of the coupling f_{Z_c} [28]. Therefore, one can neglect two-meson reducible terms and use in $\Pi_{\mu\nu}^{\text{Phys}}(p)$ single-pole zero-width approximation, as it has been done in Eq. (3).

To simplify the correlation function $\Pi_{\mu\nu}^{\text{Phys}}(p)$ and express it in terms of the tetraquark's mass and current coupling, we use the matrix element

$$\langle 0 | J_\mu | X_{AV}(p, \varepsilon) \rangle = f m \varepsilon_\mu, \quad (5)$$

and recast $\Pi_{\mu\nu}^{\text{Phys}}(p)$ into the following form

$$\Pi_{\mu\nu}^{\text{Phys}}(p) = \frac{f^2 m^2}{m^2 - p^2} \left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2} \right) + \dots \quad (6)$$

The function $\Pi_{\mu\nu}^{\text{Phys}}(p)$ has two Lorentz structures determined by $g_{\mu\nu}$ and $p_\mu p_\nu$. One of them can be chosen to continue sum rule analysis. We work with the structure proportional to $g_{\mu\nu}$ and corresponding invariant amplitude $\Pi^{\text{Phys}}(p^2)$. Advantage of this structure is that it is formed due to contributions of only spin-1 particles, and is free of any contaminations.

The QCD side of the sum rules $\Pi_{\mu\nu}^{\text{OPE}}(p)$ should be computed in the operator product expansion with some accuracy. To this end, we substitute into $\Pi_{\mu\nu}(p)$ explicit expression of the current $J_\mu(x)$, contract relevant quark fields, and replace contractions by appropriate propagators. These operations lead to the expression

$$\Pi_{\mu\nu}^{\text{OPE}}(p) = i \int d^4x e^{ipx} \epsilon \tilde{c} \epsilon' \tilde{c}' \text{Tr} \left[\gamma_5 \tilde{S}_u^{bb'}(x) \times \gamma_5 S_d^{cc'}(x) \right] \text{Tr} \left[\gamma_\mu \tilde{S}_s^{e'e}(-x) \gamma_\nu S_c^{d'd}(-x) \right], \quad (7)$$

where

$$\tilde{S}_q(x) = C S_q^T(x) C, \quad (8)$$

and $\epsilon' \tilde{c}' = \epsilon_{a'b'c'} \epsilon_{a'd'e'}$. Here, $S_c(x)$ and $S_{u(s,d)}(x)$ are the heavy c - and light $u(s, d)$ -quark propagators, respectively: Their explicit expressions are presented in Appendix (see, also Ref. [29]).

The function $\Pi_{\mu\nu}^{\text{OPE}}(p)$ is a sum of components proportional to $g_{\mu\nu}$ and $p_\mu p_\nu$. We choose the invariant amplitude $\Pi^{\text{OPE}}(p^2)$ corresponding to structure $\sim g_{\mu\nu}$, and use it to derive sum rules for m and f , which read

$$m^2 = \frac{\Pi'(M^2, s_0)}{\Pi(M^2, s_0)}, \quad (9)$$

and

$$f^2 = \frac{e^{m^2/M^2}}{m^2} \Pi(M^2, s_0). \quad (10)$$

In expressions above, $\Pi(M^2, s_0)$ is the Borel transformed and subtracted invariant amplitude $\Pi^{\text{OPE}}(p^2)$, and $\Pi'(M^2, s_0) = d\Pi(M^2, s_0)/d(-1/M^2)$.

Computing the function $\Pi(M^2, s_0)$ and fixing of regions for parameters M^2 and s_0 are next problems in our study of m and f . Calculations prove that $\Pi(M^2, s_0)$ has the form

$$\Pi(M^2, s_0) = \int_{\mathcal{M}^2}^{s_0} ds \rho^{\text{OPE}}(s) e^{-s/M^2} + \Pi(M^2), \quad (11)$$

where $\mathcal{M} = m_c + m_s$. In the present paper, we neglect the masses of u and d quarks, as well as set $m_s^2 = 0$ saving, at the same time, terms $\sim m_s$. The spectral density $\rho^{\text{OPE}}(s)$ is found as an imaginary part of the function $\Pi^{\text{OPE}}(p^2)$. Borel transformation some of terms are computed directly from expression of $\Pi_{\mu\nu}^{\text{OPE}}(p)$: They form the second component $\Pi(M^2)$ in Eq. (11). Our analysis includes contributions of different quark, gluon and mixed vacuum condensates up to dimension 10. Full analytical expressions of $\rho^{\text{OPE}}(s)$ and $\Pi(M^2)$ are written down in Appendix.

The vacuum condensates, that enter to sum rules Eqs. (9) and (10), are universal quantities obtained from analysis of various hadronic processes [16, 17, 30, 31]. Below, we list their values used in our numerical computations

$$\begin{aligned} \langle \bar{q}q \rangle &= -(0.24 \pm 0.01)^3 \text{ GeV}^3, \quad \langle \bar{s}s \rangle = (0.8 \pm 0.01) \langle \bar{q}q \rangle, \\ \langle \bar{q}g_s\sigma Gq \rangle &= m_0^2 \langle \bar{q}q \rangle, \quad \langle \bar{s}g_s\sigma Gs \rangle = m_0^2 \langle \bar{s}s \rangle, \\ m_0^2 &= (0.8 \pm 0.2) \text{ GeV}^2, \\ \left\langle \frac{\alpha_s G^2}{\pi} \right\rangle &= (0.012 \pm 0.004) \text{ GeV}^4, \\ \langle g_s^3 G^3 \rangle &= (0.57 \pm 0.29) \text{ GeV}^6, \quad m_c = 1.27 \pm 0.02 \text{ GeV}, \\ m_s &= 93_{-5}^{+11} \text{ MeV}. \end{aligned} \quad (12)$$

As is seen, the vacuum condensate of strange quarks differs from $\langle 0|\bar{q}q|0 \rangle$ [30]. The mixed condensates $\langle \bar{q}g_s\sigma Gq \rangle$ and $\langle \bar{s}g_s\sigma Gs \rangle$ are expressed using the corresponding quark condensates and parameter m_0^2 , numerical value of which was extracted from analysis of baryonic resonances [31]. For the gluon condensate $\langle g_s^3 G^3 \rangle$, we employ the estimate given in Ref. [32]. This list also contains the masses of c and s quarks from Ref. [33] in the $\overline{\text{MS}}$ -scheme.

Another problem is a choice of working windows for parameters M^2 and s_0 . They are fixed in such a way that to meet constraints imposed on $\Pi(M^2, s_0)$ by a pole contribution (PC) and convergence of the operator product expansion. These constraints can be quantified by means of the following expressions

$$\text{PC} = \frac{\Pi(M^2, s_0)}{\Pi(M^2, \infty)}, \quad (13)$$

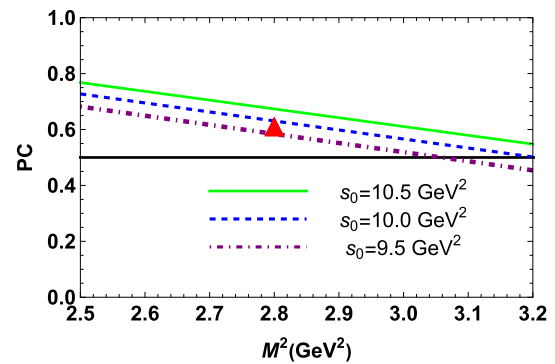


Fig. 1 The pole contribution PC as a function of the Borel parameter M^2 at different s_0 . The horizontal black line limits a region $\text{PC} = 0.5$. The red triangle marks the point, where the mass m of X_{AV} has effectively been computed

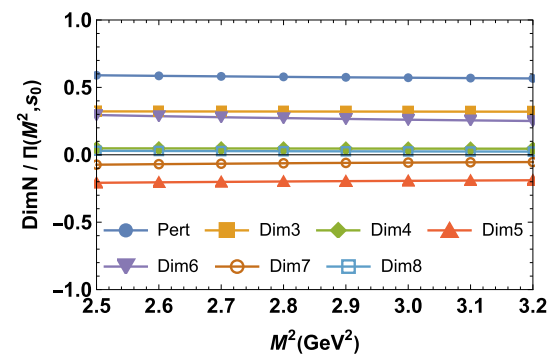


Fig. 2 Different contributions to $\Pi(M^2, s_0)$ as functions of M^2 . Dimension-9 and 10 terms are very small and not shown in the plot. All curves have been calculated at $s_0 = 10 \text{ GeV}^2$

and

$$R(M^2) = \frac{\Pi^{\text{DimN}}(M^2, s_0)}{\Pi(M^2, s_0)}, \quad (14)$$

where $\Pi^{\text{DimN}}(M^2, s_0)$ is a sum of $\text{DimN} \equiv \text{Dim}(8+9+10)$ terms. In what follows, we require fulfilment of the restrictions

$$\text{PC} \geq 0.5, \quad R(M^2) \leq 0.05. \quad (15)$$

The PC and $R(M^2)$ are employed to fix the higher and lower limits of the Borel parameter M^2 , respectively. These two values determine boundaries of the region where M^2 can be varied. Calculations show that intervals

$$M^2 \in [2.5, 3.2] \text{ GeV}^2, \quad s_0 \in [9.5, 10.5] \text{ GeV}^2, \quad (16)$$

are appropriate regions for the parameters M^2 and s_0 , and comply with limits on PC and convergence of OPE. Thus, at $M_{\text{max}}^2 = 3.2 \text{ GeV}^2$ on average in s_0 the pole contribution is 0.51, whereas at $M_{\text{min}}^2 = 2.5 \text{ GeV}^2$ it becomes equal to 0.73. To visualize dynamics of the pole contribution when varying the Borel parameter, we plot PC as a function of M^2 at different s_0 in Fig. 1. One can see, that except for a

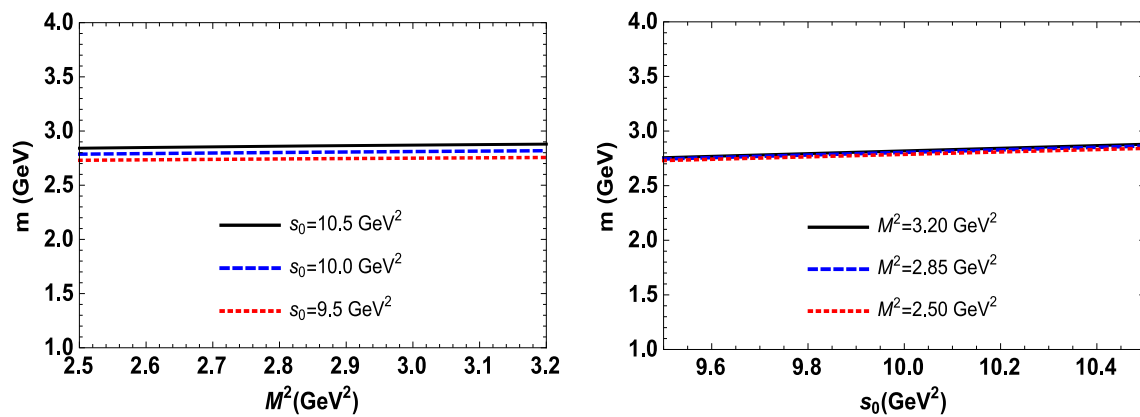


Fig. 3 Mass m of the tetraquark X_{AV} as a function of the Borel parameter M^2 (left), and the continuum threshold parameter s_0 (right)

small region $M^2 \geq 3.1 \text{ GeV}^2$ at $s_0 = 9.5 \text{ GeV}^2$ the pole contribution exceeds 0.5. On average in s_0 , the condition $\text{PC} \geq 0.5$ is fulfilled in the whole working region Eq. (16).

To be convinced in convergence of OPE, we calculate $R(M_{\min}^2)$ at the minimum point $M_{\min}^2 = 2.5 \text{ GeV}^2$, and get $R(2.5 \text{ GeV}^2) \approx 0.027$ in accordance with constraint from Eq. (15). Results of more detailed analysis are depicted in Fig. 2. In this figure, we show the perturbative and nonperturbative components of the correlation function $\Pi(M^2, s_0)$: A prevalence of the perturbative contribution to $\Pi(M^2, s_0)$ over nonperturbative one is evident. Without regard for some higher dimensional terms, the nonperturbative contributions reduce by increasing the dimensions of the corresponding operators.

The region for s_0 has to meet constraints coming from dominance of PC and convergence of OPE. Self-consistency of performed analysis can be checked by comparing the parameter $\sqrt{s_0}$ and the X_{AV} tetraquark's mass extracted from the sum rule: Evidently, an inequality $m < \sqrt{s_0}$ should be satisfied. Additionally, $\sqrt{s_0}$ bears information on a mass m^* of the first radial excitation of the tetraquark X_{AV} , therefore the restriction $m^* \geq \sqrt{s_0}$ provides low limit for m^* .

We extract the mass m and coupling f by computing them at different M^2 and s_0 , and finding their mean values averaged over the regions Eq. (16). Our predictions for m and f read

$$\begin{aligned} m &= (2800 \pm 75) \text{ MeV}, \\ f &= (2.42 \pm 0.30) \times 10^{-3} \text{ GeV}^4. \end{aligned} \quad (17)$$

The results in Eq. (17) effectively correspond to sum rules' predictions at approximately middle point of the regions in Eq. (16), i.e., to predictions at the point $M^2 = 2.8 \text{ GeV}^2$ and $s_0 = 10 \text{ GeV}^2$, where the pole contribution is $\text{PC} \approx 0.62$. This fact guarantees a dominance of the pole contribution in extracted parameters m and f .

In Fig. 3, we depict m as functions of M^2 and s_0 , in which is seen its dependence on the Borel and continuum subtraction parameters. Strictly speaking, physical quantities should

not depend on M^2 , but computations demonstrate that such effects, nevertheless, exist. Therefore, in a chosen region for M^2 this dependence should be minimal. Due to a functional form of the sum rule for the mass Eq. (9) given as the ratio of correlation functions, variation of m in the region for M^2 is mild. There is also dependence on the parameter s_0 which contains information about the lower limit for the mass of the excited tetraquark.

The pseudoscalar tetraquark X_{PS} and its parameters have been explored by the manner described just above. Here, we model X_{PS} as a tetraquark built of the pseudoscalar diquark $u^T C d$ and scalar antidiquark $\bar{c} \gamma_5 C \bar{s}^T$. The relevant interpolating current $J_{PS}(x)$ is determined by the expression

$$J_{PS}(x) = \epsilon \bar{u}_b^T(x) C d_c(x) \bar{c}_d(x) \gamma_5 C \bar{s}_e^T(x), \quad (18)$$

and belongs to the antitriplet–triplet representation of the color group $SU_c(3)$.

The physical side of the sum rule in this case has relatively simple form

$$\tilde{\Pi}^{\text{Phys}}(p) = \frac{f_{PS}^2 m_{PS}^4}{(m_c + m_s)^2 (m_{PS}^2 - p^2)} + \dots, \quad (19)$$

where m_{PS} and f_{PS} are the mass and current coupling of the tetraquark X_{PS} , respectively. To derive $\tilde{\Pi}^{\text{Phys}}(p)$, we have used the matrix element of the pseudoscalar particle X_{PS}

$$\langle 0 | J_{PS} | X_{PS}(p) \rangle = \frac{f_{PS} m_{PS}^2}{m_c + m_s}. \quad (20)$$

The function $\tilde{\Pi}^{\text{Phys}}(p)$ has trivial Lorentz structure proportional to I , therefore the invariant amplitude $\tilde{\Pi}^{\text{Phys}}(p^2)$ is equal to r.h.s. of Eq. (19).

The QCD side of new sum rules is given by the formula

$$\begin{aligned} \tilde{\Pi}^{\text{OPE}}(p) &= i \int d^4 x e^{ipx} \epsilon \bar{u} \epsilon' \bar{c} \epsilon' \text{Tr} \left[\tilde{S}_u^{bb'}(x) S_d^{cc'}(x) \right] \\ &\quad \times \text{Tr} \left[\gamma_5 \tilde{S}_s^{e'e}(-x) \gamma_5 S_c^{d'd}(-x) \right]. \end{aligned} \quad (21)$$

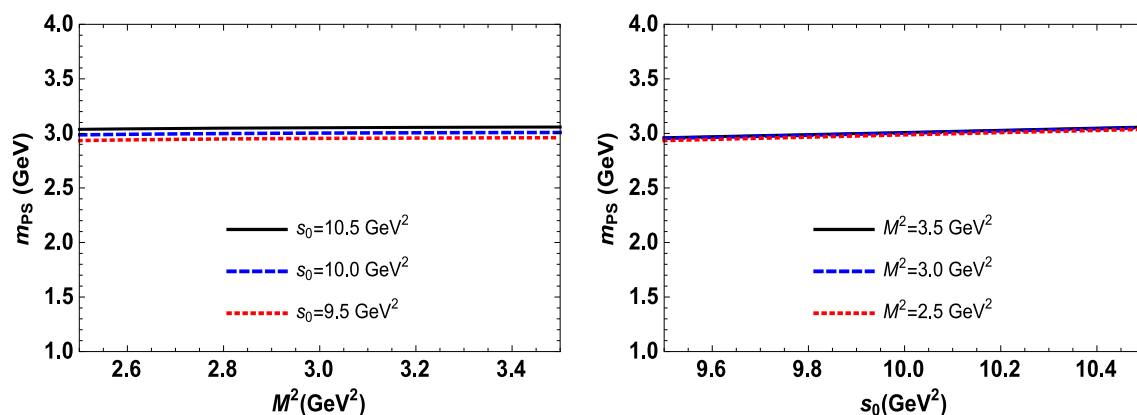


Fig. 4 Dependence of the mass m_{PS} on the Borel parameter M^2 (left), and on the continuum threshold parameter s_0 (right)

The spectroscopic parameters of X_{PS} can be obtained from Eqs. (9) and (10) after replacement $\Pi(M^2, s_0) \rightarrow \tilde{\Pi}(M^2, s_0)$. Performed calculations yield

$$\begin{aligned} m_{PS} &= (3000 \pm 60) \text{ MeV}, \\ f_{PS} &= (8.69 \pm 1.54) \times 10^{-4} \text{ GeV}^4. \end{aligned} \quad (22)$$

The Borel and continuum subtraction parameters M^2 and s_0 used to extract m_{PS} and f_{PS} are given by Eq. (23)

$$M^2 \in [2.5, 3.5] \text{ GeV}^2, \quad s_0 \in [9.5, 10.5] \text{ GeV}^2. \quad (23)$$

In these regions the PC changes inside limits

$$0.74 \geq \text{PC} \geq 0.50. \quad (24)$$

Dependence of m_{PS} on the parameters M^2 and s_0 is shown in Fig. 4. In the left panel one can see a relatively stable nature of m_{PS} under variation of M^2 .

3 Decays of the axial-vector tetraquark X_{AV}

The mass and spin-parity of the tetraquark X_{AV} allow us to classify its decay channels. We restrict ourselves by analysis of S -wave decay channels of X_{AV} which are $X_{AV} \rightarrow D^*(2010)^- K^+$, $\bar{D}^*(2007)^0 K^0$, $D^- K^*(892)^+$, and $\bar{D}^0 K^*(892)^0$. The full width of the axial-vector state X_{AV} is estimated by including into analysis namely these channels.

We are going to provide rather detailed information about computation of a partial width of the decay $X_{AV} \rightarrow D^*(2010)^- K^+$, and outline important steps in analyses of other processes. A quantity to be extracted from a sum rule is the strong coupling g_1 of particles at the vertex $X_{AV} D^*(2010)^- K^+$. This coupling is defined in terms of the on-mass-shell matrix element

$$\langle D^*(p) K(q) | X_{AV}(p') \rangle = g_1 [(p \cdot p') (\varepsilon^* \cdot \varepsilon') - (p \cdot \varepsilon') (p' \cdot \varepsilon^*)], \quad (25)$$

where the mesons K^+ and $D^*(2010)^-$ are denoted as K and D^* , respectively. Here, p' , p and q are four-momenta of the tetraquark X_{AV} , and mesons D^* and K , and ε'_ν and ε^*_μ are the polarization vectors of the particles X_{AV} and D^* .

In the framework of the LCSR method the coupling g_1 can be obtained from the correlation function

$$\Pi_{\mu\nu}(p, q) = i \int d^4x e^{ipx} \langle K(q) | T \{ J_\mu^{D^*}(x) J_\nu^\dagger(0) \} | 0 \rangle, \quad (26)$$

with $J_\nu(x)$ being the current for the tetraquark X_{AV} from Eq. (2). The interpolating current for the meson $D^*(2010)^-$ is abbreviated in Eq. (26) as $J_\mu^{D^*}(x)$, and defined by the expression

$$J_\mu^{D^*}(x) = \bar{c}_j(x) \gamma_\mu d_j(x), \quad (27)$$

where j is the color index.

The main contribution to the correlation function $\Pi_{\mu\nu}(p, q)$ comes from a term with poles at p^2 and $p'^2 = (p + q)^2$. This term is given by the formula

$$\begin{aligned} \Pi_{\mu\nu}^{\text{Phys}}(p, q) &= g_1 \frac{f m f_{D^*} m_{D^*}}{(p^2 - m_{D^*}^2)(p'^2 - m^2)} \\ &\times \left(\frac{m^2 + m_{D^*}^2 - m_K^2}{2} g_{\mu\nu} - p_\mu p'_\nu \right) + \dots, \end{aligned} \quad (28)$$

where m_{D^*} and f_{D^*} are the mass and decay constant of the meson $D^*(2010)^-$. To derive Eq. (28), we have used Eq. (25) and the following matrix elements

$$\langle 0 | J_\mu^{D^*} | D^*(p) \rangle = f_{D^*} m_{D^*} \varepsilon_\mu, \quad \langle X_{AV}(p') | J_\nu^\dagger | 0 \rangle = f m \varepsilon'_\nu. \quad (29)$$

The term written down explicitly in Eq. (28) corresponds to contribution of ground-state particles in X_{AV} and D^* channels: effects of higher resonances and continuum states in these channels are shown by dots. The function $\Pi_{\mu\nu}^{\text{Phys}}(p, q)$ constitutes the phenomenological side of a sum rule for the coupling g_1 . It contains two terms determined by structures $g_{\mu\nu}$ and $p_\mu p'_\nu$. In our studies, we use the term $\sim g_{\mu\nu}$ and

corresponding invariant amplitude $\Pi^{\text{Phys}}(p^2, p'^2)$ which is a function of two variables p^2 and p'^2 .

The correlation function $\Pi_{\mu\nu}(p, q)$ calculated in terms of quark-gluon degrees of freedom forms the QCD side of the sum rules and is equal to

$$\Pi_{\mu\nu}^{\text{OPE}}(p, q) = i \int d^4x e^{ipx} \epsilon \tilde{\epsilon} \left[\gamma_5 \tilde{S}_d^{jc}(x) \gamma^\mu \times \tilde{S}_c^{dj}(-x) \gamma_\nu \right]_{\alpha\beta} \langle K(q) | \bar{u}_\alpha^b(0) s_\beta^e(0) | 0 \rangle, \quad (30)$$

where α and β are the spinor indices.

As is seen, besides c and d quark propagators the function $\Pi_{\mu\nu}^{\text{OPE}}(p, q)$ contains also local matrix elements of the K^+ meson, which carry spinor and color indices. We can rewrite $\langle K | \bar{u}_\alpha^b s_\beta^e | 0 \rangle$ in convenient forms by expanding $\bar{u}s$ over the full set of Dirac matrices Γ^J

$$\Gamma^J = \mathbf{1}, \gamma_5, \gamma_\mu, i\gamma_5\gamma_\mu, \sigma_{\mu\nu}/\sqrt{2}, \quad (31)$$

and projecting them onto the colorless states

$$\bar{u}_\alpha^b(0) s_\beta^a(0) \rightarrow \frac{1}{12} \delta^{ba} \Gamma_{\beta\alpha}^J [\bar{u}(0) \Gamma^J s(0)]. \quad (32)$$

Operators $\bar{u}\Gamma^J s$ sandwiched between the K meson and vacuum generate local matrix elements of the K meson, which are known and can be implemented into $\Pi_{\mu\nu}^{\text{OPE}}(p, q)$.

As usual, $\Pi_{\mu\nu}^{\text{OPE}}(p, q)$ -type correlators depend on non-local matrix elements of a final meson (for example, K meson), which are convertible to its distribution amplitudes (DAs). This is correct while one treats strong vertices of three conventional mesons in the context of the LCSR method. In the case of tetraquark-meson-meson vertices relevant correlation functions instead of DAs of a final meson contain its local matrix elements. These matrix elements are determined at the space-time point $x = 0$ and are overall normalization factors. Within the LCSR method similar behavior of correlation functions was seen in a limit $q \rightarrow 0$ of three-meson vertices, which is known as a soft-meson approximation [19]. This approximation requires adoption of additional technical tools to deal with new problems appeared in a phenomenological side of corresponding sum rules [19, 20]. It turns out that the soft limit and related technical methods can be adapted to investigate also tetraquark-meson-meson vertices [21]. It is worth to emphasize that the soft limit should be implemented in a hard part of the correlation function $\Pi_{\mu\nu}(p)$, but in matrix elements one takes into account the terms with $q^2 = m_K^2$.

The term proportional to $g_{\mu\nu}$ in $\Pi_{\mu\nu}^{\text{Phys}}(p, q)$ in the limit $q \rightarrow 0$ with some accuracy can be transformed into the expression

$$\Pi_{\mu\nu}^{\text{Phys}}(p) = g_1 \frac{f m f_{D^*} m_{D^*}}{(p^2 - \bar{m}^2)^2} \left(\bar{m}^2 - \frac{m_K^2}{2} \right) g_{\mu\nu} + \dots, \quad (33)$$

where $\bar{m}^2 = (m^2 + m_{D^*}^2)/2$. The invariant amplitude $\Pi^{\text{Phys}}(p^2)$ depends on the variable p^2 , and has a double pole at $p^2 = \bar{m}^2$. The Borel transformation of $\Pi^{\text{Phys}}(p^2)$ is given by the formula

$$\mathcal{B}\Pi^{\text{Phys}}(p^2) = g_1 f m f_{D^*} m_{D^*} \left(\bar{m}^2 - \frac{m_K^2}{2} \right) \frac{e^{-\bar{m}^2/M^2}}{M^2} + \dots \quad (34)$$

The ellipses in Eq. (34) stand not only for terms suppressed after this operation, but also for contributions which remain unsuppressed even after Borel transformation. Therefore, before performing usual subtraction procedure, one should remove these contributions from $\mathcal{B}\Pi^{\text{Phys}}(p^2)$. To this end, we have to apply the operator

$$\mathcal{P}(M^2, m^2) = \left(1 - M^2 \frac{d}{dM^2} \right) M^2 e^{m^2/M^2}, \quad (35)$$

to both sides of a sum rule equality [19, 20], and subtract conventional terms in a usual way.

Then the sum rule for the strong coupling g_1 reads

$$g_1 = \frac{2}{f m f_{D^*} m_{D^*} (2\bar{m}^2 - m_K^2)} \mathcal{P}(M^2, \bar{m}^2) \Pi^{\text{OPE}}(M^2, s_0), \quad (36)$$

where $\Pi^{\text{OPE}}(M^2, s_0)$ is the Borel transformed and subtracted invariant amplitude $\Pi^{\text{OPE}}(p^2)$ that corresponds to the structure $g_{\mu\nu}$ in $\Pi_{\mu\nu}^{\text{OPE}}(p, q)$.

To finish calculation of the strong coupling g_1 , we need to specify local matrix elements of the K meson which contribute to the function $\Pi_{\mu\nu}^{\text{OPE}}(p, q)$. Details of calculations necessary to find $\Pi_{\mu\nu}^{\text{OPE}}(p, q)$ in the soft limit were presented in Ref. [21], therefore we skip further features of relevant analysis and provide only final expressions. First of all, our computations demonstrate that in the soft-meson approximation the correlator $\Pi_{\mu\nu}^{\text{OPE}}(p, q = 0)$ receives contribution from the matrix element

$$\langle 0 | \bar{u} i \gamma_5 s | K \rangle = \frac{f_K m_K^2}{m_s}, \quad (37)$$

with m_K and f_K being the mass and decay constant of the K^+ meson.

The Borel transformed and subtracted correlation function $\Pi^{\text{OPE}}(M^2, s_0)$ is calculated by taking into account condensates up to dimension 9 and given below

$$\Pi^{\text{OPE}}(M^2, s_0) = \frac{\mu_K}{48\pi^2} \int_{\mathcal{M}^2}^{s_0} \frac{ds (m_c^2 - s)^2 (m_c^2 + 2s)}{s^2} e^{-s/M^2} + \mu_K m_c \mathcal{F}^{\text{non-pert.}}(M^2), \quad (38)$$

Table 1 Masses and decay constants of the mesons D , D^* , D_0^* , K , and K^* , which have been used in numerical computations

Quantity	Value (in MeV units)
$m_{D^*} = m[D^*(2010)^-]$	2010.26 ± 0.05
$m_K = m[K^+]$	493.677 ± 0.016
$m_1 = m[\bar{D}^*(2007)^0]$	2006.85 ± 0.05
$m_2 = m[K^0]$	497.611 ± 0.013
$m_3 = m[D^-]$	1869.66 ± 0.05
$m_4 = m[K^*(892)^+]$	891.67 ± 0.26
$m_5 = m[\bar{D}^0]$	1864.84 ± 0.05
$m_6 = m[K^*(892)^0]$	895.5 ± 0.8
$m_7 = m[\bar{D}_0^*(2300)^0]$	2343 ± 10
f_K	155.7 ± 0.3
f_{K^*}	204 ± 0.3
f_D	212.6 ± 0.7
f_{D^*}	263 ± 21
$f_{D_0^*}$	373 ± 19

where $\mu_K = f_K m_K^2 / m_s$. In expression above, the nonperturbative function $\mathcal{F}^{\text{non-pert.}}(M^2)$ is determined by the formula

$$\begin{aligned} \mathcal{F}^{\text{non-pert.}}(M^2) = & -\frac{\langle \bar{d}d \rangle}{6} e^{-m_c^2/M^2} + \frac{\langle \frac{\alpha_s G^2}{\pi} \rangle m_c}{144M^4} \\ & \times \int_0^1 \frac{dx [m_c^2 + M^2 x(1-x)]}{x^3(1-x)^3} e^{-m_c^2/[M^2 x(1-x)]} \\ & - \frac{\langle \bar{d}g\sigma Gd \rangle m_c^2}{24M^4} e^{-m_c^2/M^2} + \langle \frac{\alpha_s G^2}{\pi} \rangle \langle \bar{d}d \rangle \\ & \times \frac{(m_c^2 + 3M^2)\pi^2}{108M^6} e^{-m_c^2/M^2} - \langle \frac{\alpha_s G^2}{\pi} \rangle \langle \bar{d}g\sigma Gd \rangle \\ & \times \frac{(5m_c^4 + 24m_c^2 M^2 + 6M^4)\pi^2}{432M^{10}} e^{-m_c^2/M^2}. \end{aligned} \quad (39)$$

Partial width of the process $X_{AV} \rightarrow D^*(2010)^- K^+$ can be obtained by employing the following expression

$$\Gamma[X_{AV} \rightarrow D^*(2010)^- K^+] = \frac{g_1^2 m_{D^*}^2 \lambda}{24\pi} \left(3 + \frac{2\lambda^2}{m_{D^*}^2} \right), \quad (40)$$

where $\lambda = \lambda(m, m_{D^*}, m_K)$ and

$$\begin{aligned} \lambda(a, b, c) = & \frac{1}{2a} \left[a^4 + b^4 + c^4 \right. \\ & \left. - 2(a^2 b^2 + a^2 c^2 + b^2 c^2) \right]^{1/2}. \end{aligned} \quad (41)$$

The sum rule for the strong coupling g_1 contains different vacuum condensates, numerical values of which have been collected in Eq. (12). Apart from that, the Eq. (36) depends on the spectroscopic parameters of particles involved into decay process. The mass and current coupling of the tetraquark X_{AV} have been calculated in the current work. The masses

and decay constants of the mesons $D^*(2010)^-$ and K^+ are collected in Table 1. This table contains also spectroscopic parameters of other mesons which appear at final stages of different decay channels. For masses all of the mesons and decay constants of K , K^* and D mesons, we use information from Ref. [33]. The decay constant f_{D^*} of the vector mesons $D^*(2010)^-$ and $D^*(2007)^0$, and decay constant $f_{D_0^*}$ of the scalar meson $\bar{D}_0^*(2300)^0$ are borrowed from Ref. [34].

The Borel and continuum subtraction parameters M^2 and s_0 required for calculation of the coupling g_1 are chosen in accordance with Eq. (16). Numerical computations yield

$$g_1 = (3.45 \pm 0.52) \times 10^{-1} \text{ GeV}^{-1}. \quad (42)$$

Then it is not difficult to find

$$\Gamma[X_{AV} \rightarrow D^*(2010)^- K^+] = (10.6 \pm 3.4) \text{ MeV}. \quad (43)$$

The decay of the tetraquark X_{AV} to a meson pair $\bar{D}^*(2007)^0 K^0$ is another process with K meson in the final state. This process is a “neutral” version of the first channel differences being encoded in masses m_1 and m_2 of mesons $\bar{D}^*(2007)^0$ and K^0 , respectively. Treatment of this decay mode does not differ from analysis described above. Therefore, we provide final results for the coupling g_2

$$g_2 = (3.67 \pm 0.55) \times 10^{-1} \text{ GeV}^{-1}, \quad (44)$$

and partial width of the process

$$\Gamma[X_{AV} \rightarrow \bar{D}^*(2007)^0 K^0] = (11.9 \pm 3.8) \text{ MeV}. \quad (45)$$

The remaining two decay channels $X_{AV} \rightarrow D^- K^*(892)^+$ and $X_{AV} \rightarrow \bar{D}^0 K^*(892)^0$ have been explored by a similar manner. Let us consider, for instance, the process $X_{AV} \rightarrow D^- K^*(892)^+$. The strong coupling g_3 that corresponds to the vertex $X_{AV} D^- K^*(892)^+$ is defined by the matrix element

$$\langle D^-(p) K^*(q) | X_{AV}(p') \rangle = g_3 \left[(q \cdot p') (\epsilon^* \cdot \epsilon') - (q \cdot \epsilon') (p' \cdot \epsilon^*) \right], \quad (46)$$

where ϵ_μ^* is polarization vector of the meson $K^*(892)^+$. The correlation function that allows us to extract g_3 is

$$\Pi_V(p, q) = i \int d^4 x e^{ipx} \langle K^*(q) | T \{ J^D(x) J_V^\dagger(0) \} | 0 \rangle, \quad (47)$$

with $J^D(x)$ being the interpolating current for the pseudoscalar meson D^-

$$J^D(x) = \bar{c}_j(x) i \gamma_5 d_j(x). \quad (48)$$

The main term that contributes to this correlation function and determines phenomenological side of a sum rule for g_3 has the following form

Table 2 Decay channels of the tetraquarks X_{AV} and X_{PS} , strong couplings g_i , G_j and partial widths Γ_{AV}^i , Γ_{PS}^j . The star-marked coupling G_1 has a dimension GeV^{-1}

i	Channels	g_i (GeV^{-1})	Γ_{AV}^i (MeV)
1	$X_{AV} \rightarrow D^*(2010)^- K^+$	$(3.45 \pm 0.52) \times 10^{-1}$	10.6 ± 3.4
2	$X_{AV} \rightarrow \bar{D}^*(2007)^0 K^0$	$(3.67 \pm 0.55) \times 10^{-1}$	11.9 ± 3.8
3	$X_{AV} \rightarrow D^- K^*(892)^+$	1.52 ± 0.23	17.1 ± 5.7
4	$X_{AV} \rightarrow \bar{D}^0 K^*(892)^0$	1.55 ± 0.25	18.1 ± 6.2
j		G_j	Γ_{PS}^j (MeV)
1	$X_{PS} \rightarrow \bar{D}_0^*(2300)^0 K^0$	$(4.41 \pm 0.66) \times 10^{-1*}$	16.6 ± 5.3
2	$X_{PS} \rightarrow D^- K^*(892)^+$	1.67 ± 0.35	23.6 ± 7.6
3	$X_{PS} \rightarrow \bar{D}^0 K^*(892)^0$	1.70 ± 0.38	24.5 ± 7.9

$$\Pi_v^{\text{Phys}}(p, q) = g_3 \frac{f m f_D m_3^2}{m_c (p^2 - m_3^2) (p'^2 - m^2)} \times \left(\frac{m^2 + m_3^2 - m_4^2}{2} \epsilon_v^* + p' \cdot \epsilon^* q_v \right) + \dots, \quad (49)$$

where m_3 and m_4 are masses of D^- and $K^*(892)^+$, respectively. Here, f_D is the decay constant of the meson D^- . To find $\Pi_v^{\text{Phys}}(p, q)$, we use the matrix elements of the tetraquark X_{AV} and vertex, as well as new matrix element

$$\langle 0 | J^D(x) | D^- \rangle = \frac{f_D m_3^2}{m_c}. \quad (50)$$

The same function $\Pi_v(p, q)$ calculated in term of quark-gluon degrees of freedom gives QCD side of the sum rule

$$\Pi_v^{\text{OPE}}(p, q) = -i^2 \int d^4 x e^{ipx} \epsilon \tilde{\epsilon} \left[\gamma_5 \tilde{S}_d^{jc}(x) \gamma_5 \times \tilde{S}_c^{dj}(-x) \gamma_\nu \right]_{\alpha\beta} \langle K^*(q) | \bar{u}_\alpha^b(0) s_\beta^e(0) | 0 \rangle. \quad (51)$$

Our analysis demonstrates that in the soft-meson approximation a contribution to $\Pi_v^{\text{OPE}}(p, q)$ comes from the local matrix element of the meson $K^*(892)^+$

$$\langle 0 | \bar{s}(0) \gamma_\mu u(0) | K^* \rangle = \epsilon_\mu f_{K^*} m_4, \quad (52)$$

with f_{K^*} being its decay constant.

The sum rule for g_3 can be determined using structures proportional to ϵ_v^* in $\Pi_v^{\text{Phys}}(p, q)$ and $\Pi_v^{\text{OPE}}(p, q)$, and corresponding invariant amplitudes $\tilde{\Pi}^{\text{Phys}}(p^2)$ and $\tilde{\Pi}^{\text{OPE}}(p^2)$. The amplitude $\tilde{\Pi}^{\text{OPE}}(p^2)$ is calculated by including effects of condensates up to dimension 9. After Borel transformation and subtraction it takes the form

$$\tilde{\Pi}^{\text{OPE}}(M^2, s_0) = \frac{f_{K^*} m_4}{16\pi^2} \int_{\mathcal{M}^2}^{s_0} \frac{ds (m_c^2 - s)^2}{s} e^{-s/M^2} + m_4 m_c f_{K^*} \tilde{\mathcal{F}}^{\text{non-pert.}}(M^2). \quad (53)$$

The function $\tilde{\mathcal{F}}^{\text{non-pert.}}(M^2)$ in Eq. (53) is given by the expression

$$\tilde{\mathcal{F}}^{\text{non-pert.}}(M^2) = -\frac{\langle \bar{d}d \rangle}{6} e^{-m_c^2/M^2} + \frac{\langle \frac{\alpha_s G^2}{\pi} \rangle m_c^3}{144 M^4} \times \int_0^1 \frac{dx e^{-m_c^2/[M^2 x(1-x)]}}{x^3(x-1)^3} + \frac{\langle \bar{d}g\sigma Gd \rangle m_c^2}{24 M^4} e^{-m_c^2/M^2} - \left\langle \frac{\alpha_s G^2}{\pi} \right\rangle \langle \bar{d}d \rangle \frac{(m_c^2 + 3M^2)\pi^2}{108 M^6} e^{-m_c^2/M^2} + \left\langle \frac{\alpha_s G^2}{\pi} \right\rangle \times \langle \bar{d}g\sigma Gd \rangle \frac{(5m_c^4 + 24m_c^2 M^2 + 6M^4)\pi^2}{432 M^{10}} e^{-m_c^2/M^2}. \quad (54)$$

After manipulations described above in detail, for g_3 we get the sum rule

$$g_3 = \frac{2m_c}{f m f_D m_3^2 (2\tilde{m}^2 - m_{K^*}^2)} \mathcal{P}(M^2, \tilde{m}^2) \tilde{\Pi}^{\text{OPE}}(M^2, s_0), \quad (55)$$

where $\tilde{m}^2 = (m^2 + m_3^2)/2$.

The sum rule prediction for g_3 reads

$$g_3 = (1.52 \pm 0.23) \text{ GeV}^{-1}. \quad (56)$$

The partial width of the process $X_{AV} \rightarrow D^- K^*(892)^+$ can be computed using Eq. (40) after replacing g_1, λ , and m_{D^*} by $g_3, \tilde{\lambda}$, and m_4 , where $\tilde{\lambda} = \lambda(m^2, m_3^2, m_4^2)$. Calculations lead to the following result

$$\Gamma[X_{AV} \rightarrow D^- K^*(892)^+] = (17.1 \pm 5.7) \text{ MeV}. \quad (57)$$

Predictions for the strong coupling g_4 and partial width of the decay $X_{AV} \rightarrow \bar{D}^0 K^*(892)^0$, as well as results obtained in the present section are collected in Table 2. This information allows us to estimate full width of the tetraquark X_{AV}

$$\Gamma_{AV} = (58 \pm 10) \text{ MeV}, \quad (58)$$

which characterizes it as a resonance with a relatively narrow width.

4 Processes $X_{\text{PS}} \rightarrow \bar{D}_0^*(2300)^0 K^0$, $D^- K^*(892)^+$ and $\bar{D}^0 K^*(892)^0$

In this section, we investigate decays of the pseudoscalar tetraquark X_{PS} with the mass m_{PS} and current coupling f_{PS} which have been extracted from two-point sum rules in Sect. 2. We consider the S -wave process $X_{\text{PS}} \rightarrow \bar{D}_0^*(2300)^0 K^0$, as well as P -wave decay modes $X_{\text{PS}} \rightarrow D^- K^*(892)^+$ and $X_{\text{PS}} \rightarrow \bar{D}^0 K^*(892)^0$ of this four-quark state.

We begin our analysis from the decay $X_{\text{PS}} \rightarrow \bar{D}_0^*(2300)^0 K^0$. The coupling G_1 required to calculate partial width of this process can be defined using the on-mass-shell matrix element

$$\langle D_0^*(p) K^0(q) | X_{\text{PS}}(p') \rangle = G_1 p \cdot p'. \quad (59)$$

The correlation function, which should be considered to determine strong coupling G_1 of particles at the vertex $X_{\text{PS}} \bar{D}_0^*(2300)^0 K^0$, is given by the formula

$$\hat{\Pi}(p, q) = i \int d^4 x e^{ipx} \langle K^0(q) | T \{ J^{D_0^*}(x) J_{\text{PS}}^\dagger(0) \} | 0 \rangle, \quad (60)$$

where D_0^* stands for meson $\bar{D}_0^*(2300)^0$. Here, $J_{\text{PS}}(x)$ and $J^{D_0^*}(x)$ are interpolating currents for the tetraquark X_{PS} [see, Eq. (18)], and for the scalar meson $\bar{D}_0^*(2300)^0$. The latter is defined by the expression

$$J^{D_0^*}(x) = \bar{c}_j(x) d_j(x). \quad (61)$$

A contribution to the correlation function $\hat{\Pi}^{\text{Phys}}(p, q)$ with poles at p^2 and $p'^2 = (p + q)^2$ comes from the term

$$\begin{aligned} \hat{\Pi}^{\text{Phys}}(p, q) = G_1 & \frac{f_{\text{PS}} m_{\text{PS}}^2 f_{D_0^*} m_7}{(m_c + m_s)(p^2 - m_7^2)(p'^2 - m_{\text{PS}}^2)} \\ & \times \frac{m_{\text{PS}}^2 + m_7^2 - m_2^2}{2} + \dots \end{aligned} \quad (62)$$

Here, m_7 and $f_{D_0^*}$ are the mass and decay constant of the meson $\bar{D}_0^*(2300)^0$, respectively. In order to find Eq. (62), we employ Eqs. (59) and (20), as well as the matrix element

$$\langle 0 | J^{D_0^*} | D_0^*(p) \rangle = f_{D_0^*} m_7. \quad (63)$$

The QCD side of the sum rule $\hat{\Pi}^{\text{OPE}}(p, q)$ has the following form

$$\begin{aligned} \hat{\Pi}^{\text{OPE}}(p, q) = i \int d^4 x e^{ipx} \epsilon \tilde{\epsilon} \left[\tilde{S}_u^{jb}(x) \tilde{S}_c^{dj}(-x) \gamma_5 \right]_{\alpha\beta} \\ \times \langle K^0(q) | \bar{d}_\alpha^c(0) s_\beta^c(0) | 0 \rangle. \end{aligned} \quad (64)$$

The functions $\hat{\Pi}^{\text{Phys}}(p, q)$ and $\hat{\Pi}^{\text{OPE}}(p, q)$ have trivial Lorentz structures $\sim I$, therefore both of them contain only one invariant amplitude. The amplitude $\hat{\Pi}^{\text{OPE}}(p^2)$ is calculated with dimension-9 accuracy and given by the following

expression

$$\begin{aligned} \hat{\Pi}^{\text{OPE}}(M^2, s_0) = & \frac{\mu_{K^0}}{16\pi^2} \int_{\mathcal{M}^2}^{s_0} \frac{ds(m_c^2 - s)^2}{s} e^{-s/M^2} \\ & + \mu_{K^0} m_c \hat{\mathcal{F}}^{\text{non-pert.}}(M^2), \end{aligned} \quad (65)$$

where the function $\hat{\mathcal{F}}^{\text{non-pert.}}(M^2)$ is determined by the formula

$$\begin{aligned} \hat{\mathcal{F}}^{\text{non-pert.}}(M^2) = & \frac{\langle \bar{u}u \rangle}{16} e^{-m_c^2/M^2} - \frac{\langle \frac{\alpha_s G^2}{\pi} \rangle m_c^3}{144M^4} \\ & \times \int_0^1 \frac{dx e^{-m_c^2/[M^2 x(1-x)]}}{x^3(x-1)^3} - \frac{\langle \bar{u}g\sigma Gu \rangle m_c^2}{24M^4} e^{-m_c^2/M^2} \\ & + \left\langle \frac{\alpha_s G^2}{\pi} \right\rangle \langle \bar{u}u \rangle \frac{(m_c^2 + 3M^2)\pi^2}{108M^6} e^{-m_c^2/M^2} - \left\langle \frac{\alpha_s G^2}{\pi} \right\rangle \\ & \times \langle \bar{u}g\sigma Gu \rangle \frac{(5m_c^4 + 24m_c^2 M^2 + 6M^4)\pi^2}{432M^{10}} e^{-m_c^2/M^2}, \end{aligned} \quad (66)$$

and $\mu_{K^0} = m_2^2 f_K / m_s$.

In the soft-meson approximation the coupling G_1 can be found by means of the sum rule

$$\begin{aligned} G_1 = & \frac{2(m_c + m_s)}{f_{\text{PS}} m_{\text{PS}}^2 f_{D_0^*} m_7 (2\hat{m}^2 - m_K^2)} \\ & \times \mathcal{P}(M^2, \hat{m}^2) \hat{\Pi}^{\text{OPE}}(M^2, s_0), \end{aligned} \quad (67)$$

where $\hat{m}^2 = (m_{\text{PS}}^2 + m_7^2)/2$.

The width of the decay $X_{\text{PS}} \rightarrow \bar{D}_0^*(2300)^0 K^0$ is found by utilizing the formula

$$\Gamma[X_{\text{PS}} \rightarrow \bar{D}_0^*(2300)^0 K^0] = G_1^2 \frac{m_7^2 \hat{\lambda}}{8\pi} \left(1 + \frac{\hat{\lambda}^2}{m_7^2} \right), \quad (68)$$

in which $\hat{\lambda} = \lambda(m_{\text{PS}}, m_7, m_2)$. Our computations for the coupling G_1 and partial width of the process yield

$$G_1 = (4.41 \pm 0.66) \times 10^{-1} \text{ GeV}^{-1}, \quad (69)$$

and

$$\Gamma[X_{\text{PS}} \rightarrow \bar{D}_0^*(2300)^0 K^0] = (16.6 \pm 5.3) \text{ MeV}. \quad (70)$$

The coupling G_2 that describes strong interaction of particles at the vertex $X_{\text{PS}} D^- K^*(892)^+$ and determines partial width of the channel $X_{\text{PS}} \rightarrow D^- K^*(892)^+$ is defined by the matrix element

$$\langle D^-(p) K^*(q) | X_{\text{PS}}(p') \rangle = G_2 p \cdot \epsilon^*. \quad (71)$$

The sum rule for G_2 is obtained from the correlation function

$$\Pi'(p, q) = i \int d^4 x e^{ipx} \langle K^*(q) | T \{ J^D(x) J_{\text{PS}}^\dagger(0) \} | 0 \rangle. \quad (72)$$

In terms of involved particles' physical parameters this function has the form

$$\Pi'^{\text{Phys}}(p, q) = G_2 \frac{f_{\text{PS}} m_{\text{PS}}^2}{(m_c + m_s)(p^2 - m_{\text{PS}}^2)} \times \frac{f_D m_3^2}{m_c(p^2 - m_3^2)} p \cdot \epsilon^* + \dots \quad (73)$$

To extract this expression, we use the matrix elements from Eqs. (50) and (52), as well as one defined by Eq. (71).

The correlation function $\Pi'(p, q)$ calculated using quark-gluon degrees of freedom fixes the QCD side of the sum rule and is equal to

$$\Pi'^{\text{OPE}}(p, q) = - \int d^4x e^{ipx} \epsilon \tilde{\epsilon} \left[\tilde{S}_d^{jc}(x) \gamma_5 \tilde{S}_c^{dj}(-x) \gamma_5 \right]_{\alpha\beta} \times \langle K^*(q) | \bar{u}_\alpha^b(0) s_\beta^c(0) | 0 \rangle. \quad (74)$$

The sum rule for the strong coupling G_2 can be obtained by employing standard manipulations. The width of the process $X_{\text{PS}} \rightarrow D^- K^*(892)^+$ is calculated by means of the expression

$$\Gamma[X_{\text{PS}} \rightarrow D^- K^*(892)^+] = G_2^2 \frac{\lambda'^3}{8\pi m_4^2}, \quad (75)$$

where $\lambda' = \lambda(m_{\text{PS}}, m_3, m_4)$.

For the coupling G_2 numerical computations give

$$G_2 = 1.67 \pm 0.35, \quad (76)$$

and the partial width of the decay under analysis equals to

$$\Gamma[X_{\text{PS}} \rightarrow D^- K^*(892)^+] = (23.6 \pm 7.6) \text{ MeV}. \quad (77)$$

The decay channel $X_{\text{PS}} \rightarrow \bar{D}^0 K^*(892)^0$ of the tetraquark X_{PS} which is last process considered in the present paper, can be treated in a similar way. Therefore, we refrain from further details and provide all relevant information in Table 2.

The full width of X_{PS}

$$\Gamma_{\text{PS}} = (65 \pm 12) \text{ MeV}, \quad (78)$$

does not differ considerably from the width of the axial-vector tetraquark X_{AV} .

5 Conclusions

In the current article, we have investigated the axial-vector and pseudoscalar tetraquarks X_{AV} and X_{PS} from a family of exotic mesons $[ud][\bar{c}\bar{s}]$ containing four different quark flavors. We have calculated their masses, and also estimated full widths of these states using different decay channels.

Interest to fully open-flavor structures renewed recently due to discovery of resonances $X_{0(1)}(2900)$ made by the LHCb collaboration. One of these states $X_1(2900)$ was studied in Ref. [7] as a vector tetraquark $X_V = [ud][\bar{c}\bar{s}]$. The mass and width of X_V are close to physical parameters of the resonance $X_1(2900)$ measured by LHCb, which allowed

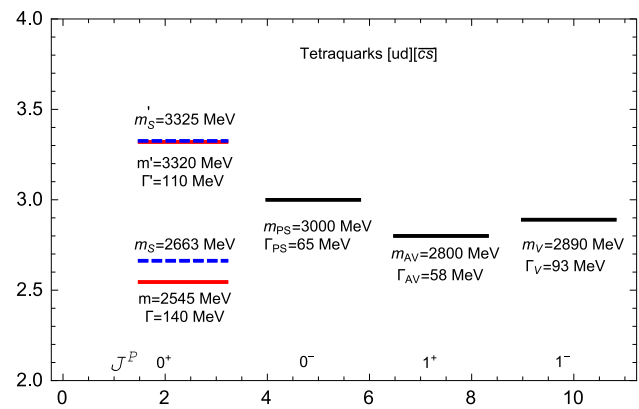


Fig. 5 Mass and width of tetraquarks $[ud][\bar{c}\bar{s}]$ with different spin-parities. The lower and upper red (solid) and blue (dashed) lines are $1S$ and $2S$ scalar states, respectively. The red lines correspond to the scalar tetraquark X_0 , whereas blue lines show parameters of X_S . The mass and width of the vector particle X_V were determined in Ref. [7]. The axial-vector and pseudoscalar states have been explored in the current article. Theoretical uncertainties of extracted observables are not shown

us to interpret X_V as a candidate to the vector resonance $X_1(2900)$.

The masses and widths of the ground-state scalar tetraquarks X_0 and X_S , and their first radial excitations were calculated in Ref. [9]. The scalar particles X_0 and X_S were modeled using axial-vector and scalar diquark–antidiquark pairs, respectively.

Information gained in our studies about tetraquarks $[ud][\bar{c}\bar{s}]$ with spin-parities $J^P = 0^+, 0^-, 1^+, 1^-$ is shown in Fig. 5. As is seen, the pseudoscalar X_{PS} state is heaviest particle in family of tetraquarks $[ud][\bar{c}\bar{s}]$, whereas light particles are scalar ones with different internal organizations. The vector and axial-vector states have comparable masses and widths.

It is interesting to consider hadronic processes, where exotic mesons X_{AV} and X_{PS} may be observed. We have noted in Sect. 1 that resonances $X_0(2900)$ and $X_1(2900)$ were discovered in the invariant $D^- K^+$ mass distribution of the process $B^+ \rightarrow D^+ D^- K^+$. For the scalar tetraquark X_0 the decay $X_0 \rightarrow D^- K^+$ is S -wave process, whereas $X_1 \rightarrow D^- K^+$ is P -wave channel for the vector particle $X_1(2900)$. The invariant mass distribution of $D^- K^*(892)^+$ mesons in exclusive decays of B^+ meson may be explored to observe tetraquarks X_{AV} and X_{PS} . In fact, decays to $D^- K^*(892)^+$ mesons are S -wave and P -wave channels for the tetraquarks X_{AV} and X_{PS} , respectively. Because branching ratios of these channels amount to 0.30 and 0.36, respectively, they may be employed to fix structures X_{AV} and X_{PS} . These and relevant problems require further theoretical and experimental studies.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: All the numerical

and mathematical data have been included in the paper and we have no other data regarding this paper.]

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Funded by SCOAP³. SCOAP³ supports the goals of the International Year of Basic Sciences for Sustainable Development.

Appendix: Quark propagators and invariant amplitude $\Pi(M^2, s_0)$

In this work, for the light quark propagator $S_q^{ab}(x)$, we use the expression

$$\begin{aligned} S_q^{ab}(x) = & i\delta_{ab} \frac{\not{x}}{2\pi^2 x^4} - \delta_{ab} \frac{m_q}{4\pi^2 x^2} - \delta_{ab} \frac{\langle \bar{q}q \rangle}{12} \\ & + i\delta_{ab} \frac{\not{x} m_q \langle \bar{q}q \rangle}{48} - \delta_{ab} \frac{x^2}{192} \langle \bar{q} g_s \sigma G q \rangle \\ & + i\delta_{ab} \frac{x^2 \not{x} m_q}{1152} \langle \bar{q} g_s \sigma G q \rangle - i \frac{g_s G_{ab}^{\alpha\beta}}{32\pi^2 x^2} [\not{x} \sigma_{\alpha\beta} + \sigma_{\alpha\beta} \not{x}] \\ & - i\delta_{ab} \frac{x^2 \not{x} g_s^2 \langle \bar{q}q \rangle^2}{7776} \\ & - \delta_{ab} \frac{x^4 \langle \bar{q}q \rangle \langle g_s^2 G^2 \rangle}{27648} + \dots \end{aligned} \quad (\text{A.1})$$

For the heavy quark $Q = c$, we employ the propagator $S_Q^{ab}(x)$

$$\begin{aligned} S_Q^{ab}(x) = & i \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} \left\{ \frac{\delta_{ab} (\not{k} + m_Q)}{k^2 - m_Q^2} \right. \\ & - \frac{g_s G_{ab}^{\alpha\beta}}{4} \frac{\sigma_{\alpha\beta} (\not{k} + m_Q) + (\not{k} + m_Q) \sigma_{\alpha\beta}}{(k^2 - m_Q^2)^2} \\ & + \frac{g_s^2 G^2}{12} \delta_{ab} m_Q \frac{k^2 + m_Q^2}{(k^2 - m_Q^2)^4} \\ & + \frac{g_s^3 G^3}{48} \delta_{ab} \frac{(\not{k} + m_Q)}{(k^2 - m_Q^2)^6} \left[\not{k} (k^2 - 3m_Q^2) \right. \\ & \left. \left. + 2m_Q (2k^2 - m_Q^2) \right] (\not{k} + m_Q) + \dots \right\}. \end{aligned} \quad (\text{A.2})$$

Above, we have used the shorthand notations

$$\begin{aligned} G_{ab}^{\alpha\beta} & \equiv G_A^{\alpha\beta} \lambda_{ab}^A / 2, \quad G^2 = G_{\alpha\beta}^A G_A^{\alpha\beta}, \quad G^3 \\ & = f^{ABC} G_{\alpha\beta}^A G^{B\beta\delta} G_\delta^{C\alpha}, \end{aligned} \quad (\text{A.3})$$

where $G_A^{\alpha\beta}$ is the gluon field strength tensor, λ^A and f^{ABC} are the Gell-Mann matrices and structure constants of the color group $SU_c(3)$, respectively. The indices A, B, C run in the range $1, 2, \dots, 8$.

The invariant amplitude $\Pi(M^2, s_0)$ obtained after the Borel transformation and subtraction is equal to

$$\Pi(M^2, s_0) = \int_{\mathcal{M}^2}^{s_0} ds \rho^{\text{OPE}}(s) e^{-s/M^2} + \Pi(M^2),$$

where the spectral density $\rho^{\text{OPE}}(s)$ and the function $\Pi(M^2)$ are determined by formulas

$$\begin{aligned} \rho^{\text{OPE}}(s) &= \rho^{\text{pert.}}(s) + \sum_{N=3}^8 \rho^{\text{DimN}}(s), \\ \Pi(M^2) &= \sum_{N=6}^{10} \Pi^{\text{DimN}}(M^2), \end{aligned} \quad (\text{A.4})$$

respectively. The components of $\rho^{\text{OPE}}(s)$ and $\Pi(M^2)$ are given by expressions

$$\begin{aligned} \rho^{\text{DimN}}(s) &= \int_0^1 d\alpha \rho^{\text{DimN}}(s, \alpha), \quad \Pi^{\text{DimN}}(M^2) \\ &= \int_0^1 d\alpha \Pi^{\text{DimN}}(M^2, \alpha), \end{aligned} \quad (\text{A.5})$$

where α is the Feynman parameter.

Below, we write down components of the spectral density $\rho^{\text{OPE}}(s)$ and function $\Pi(M^2)$ for the axial-vector tetraquark X_{AV} :

The perturbative and nonperturbative components of the spectral density $\rho^{\text{pert.}}(s, \alpha)$ and $\rho^{\text{Dim3(4,5,6,7,8)}}(s, \alpha)$ have the forms:

$$\begin{aligned} \rho^{\text{pert.}}(s, \alpha) &= \frac{\Theta(L)}{12288\pi^6(1-\alpha)^3} \left[m_c^2 - s(1-\alpha) \right]^3 \\ &\times \alpha^3 \left[m_c^2 \alpha - 16m_c m_s - 5s\alpha(1-\alpha) \right], \end{aligned} \quad (\text{A.6})$$

$$\begin{aligned} \rho^{\text{Dim3}}(s, \alpha) &= -\frac{\langle \bar{s}s \rangle \Theta(L) \alpha^2}{128\pi^4(1-\alpha)^2} \left[m_c^2 - s(1-\alpha) \right] \left[2m_c^3 \right. \\ &\left. - m_c^2 m_s(1-\alpha) - 2m_c s(1-\alpha) + 3m_s s(1-\alpha)^2 \right], \end{aligned} \quad (\text{A.7})$$

$$\begin{aligned} \rho^{\text{Dim4}}(s, \alpha) &= \frac{\langle \alpha_s G^2/\pi \rangle \Theta(L) \alpha}{55296\pi^4(1-\alpha)^3} \left[s^2 \alpha(1-\alpha)^3 (162 - 167\alpha) \right. \\ &+ 3m_c^4 \alpha (18 - 37\alpha + 21\alpha^2) \\ &- 24m_c^3 m_s (9 - 18\alpha + 9\alpha^2 + 2\alpha^3) \\ &+ 4sm_c^2 \alpha (-54 + 164\alpha - 169\alpha^2 + 59\alpha^3) \\ &\left. - 24sm_c m_s (-9 + 27\alpha - 26\alpha^2 + 5\alpha^3 + 3\alpha^4) \right], \end{aligned} \quad (\text{A.8})$$

$$\begin{aligned} \rho^{\text{Dim5}}(s, \alpha) &= -\frac{\langle \bar{s}s \rangle g_s \sigma G s \rangle \Theta(L) \alpha}{192\pi^4(1-\alpha)} \left[3m_c^3 - m_c^2 m_s(1-\alpha) \right. \\ &\left. - 3sm_c(1-\alpha) + 2sm_s(1-\alpha)^2 \right], \end{aligned} \quad (\text{A.9})$$

$$\rho^{\text{Dim6}}(s, \alpha) = -\frac{\Theta(L)}{405 \cdot 2^{12} \pi^6 (1-\alpha)^3}$$

$$\begin{aligned} & \times \left\{ (1-\alpha)^3 \left[-m_c m_s + m_c^2 \alpha - 2s\alpha(1-\alpha) \right] \right. \\ & \times (1280 \langle \bar{d}d \rangle^2 g_s^2 \pi^2 + 138240 \langle \bar{d}d \rangle \langle \bar{u}u \rangle \pi^4) \\ & - \alpha \left[1280 g_s^2 \pi^2 \langle \bar{s}s \rangle^2 (m_c^2 - 2s(1-\alpha))(\alpha-1)^3 \right. \\ & \left. \left. + 27 \langle g_s^3 G^3 \rangle m_c^2 \alpha^4 \right] \right\}, \end{aligned} \quad (\text{A.10})$$

$$\begin{aligned} \rho^{\text{Dim7}}(s, \alpha) = & -\frac{\langle \alpha_s G^2/\pi \rangle \langle \bar{s}s \rangle \Theta(L)}{1152\pi^2(1-\alpha)^2} [4m_c(2-4\alpha \\ & + 2\alpha^2 + \alpha^3) - 3m_s(1-\alpha)^3], \end{aligned} \quad (\text{A.11})$$

$$\begin{aligned} \rho^{\text{Dim8}}(s, \alpha) = & \frac{\Theta(L)}{18432\pi^2} [768 \langle \bar{u}u \rangle \langle \bar{d}g_s \sigma G d \rangle (\alpha-1) \\ & - \langle \alpha_s G^2/\pi \rangle^2 \alpha]. \end{aligned} \quad (\text{A.12})$$

Components of the function $\Pi(M^2)$ are:

$$\begin{aligned} \Pi^{\text{Dim6}}(M^2, \alpha) = & \frac{\langle g_s^3 G^3 \rangle m_c^3 \alpha^3}{184320\pi^6 M^2 (1-\alpha)^5} \\ & \times \exp \left[-\frac{m_c^2}{M^2(1-\alpha)} \right] [m_c^3 \alpha(2+\alpha) + 8m_c^2 m_s(2\alpha-1) \\ & - 16M^2 m_s(\alpha^2-1)], \end{aligned} \quad (\text{A.13})$$

$$\begin{aligned} \Pi^{\text{Dim7}}(M^2, \alpha) = & -\frac{\langle \alpha_s G^2/\pi \rangle \langle \bar{s}s \rangle m_c^3 \alpha^2}{1152M^2\pi^2(1-\alpha)^3} \\ & \times \exp \left[-\frac{m_c^2}{M^2(1-\alpha)} \right] [m_c m_s - 2M^2(1-\alpha)], \end{aligned} \quad (\text{A.14})$$

$$\begin{aligned} \Pi^{\text{Dim8}}(M^2, \alpha) = & -\frac{\langle \alpha_s G^2/\pi \rangle^2 m_c \alpha}{9216M^2\pi^2(1-\alpha)^3} \\ & \times \exp \left[-\frac{m_c^2}{M^2(1-\alpha)} \right] [2m_c^2 m_s(1-\alpha) + m_c^3 \alpha \\ & + 2M^2 m_s(\alpha^2-1)] \\ & - \frac{\langle \bar{u}u \rangle \langle \bar{d}g_s \sigma G d \rangle m_c m_s}{24\pi^2} \exp \left[-\frac{m_c^2}{M^2} \right], \end{aligned} \quad (\text{A.15})$$

$$\begin{aligned} \Pi^{\text{Dim9}}(M^2, \alpha) = & \frac{m_c \alpha}{69120M^6\pi^4(1-\alpha)^5} \\ & \times \exp \left[-\frac{m_c^2}{M^2(1-\alpha)} \right] \{ 20 \langle \alpha_s G^2/\pi \rangle \langle \bar{s}s \rangle \sigma G s \rangle M^2 \pi^2 (1-\alpha)^2 \\ & \times [m_c^3 m_s - 3m_c^2 M^2(1-\alpha) - m_c m_s M^2(1-\alpha) \\ & - 3M^4(\alpha^2-1)] + 3 \langle g_s^3 G^3 \rangle \langle \bar{s}s \rangle m_c \alpha [4m_c^3 M^2(1-2\alpha) \\ & - 16m_c M^4(1-\alpha) + 4m_c^4 m_s(2+\alpha) + m_c^2 m_s M^2(\alpha^2 \\ & + \alpha-2) - m_s M^4(2-3\alpha+\alpha^3)] \} \\ & + \frac{m_c}{62208\pi^2 M^2} \\ & \times \exp \left[-\frac{m_c^2}{M^2} \right] \{ -27 \langle \alpha_s G^2/\pi \rangle \langle \bar{s}s \rangle \sigma G s \rangle (m_c m_s - 3M^2) \\ & + 32 g_s^2 \langle \bar{s}s \rangle (\langle \bar{u}u \rangle^2 + \langle \bar{d}d \rangle^2) \\ & \times (m_c m_s - 2M^2) + 3456 \langle \bar{s}s \rangle \langle \bar{u}u \rangle \langle \bar{d}d \rangle \pi^2 (m_c m_s - 2M^2) \}, \end{aligned} \quad (\text{A.16})$$

$$\begin{aligned} \Pi^{\text{Dim10}}(M^2, \alpha) = & -\frac{m_c \langle \alpha_s G^2/\pi \rangle g_s^2}{23328M^4\pi^2(1-\alpha)^3} \\ & \times \exp \left[-\frac{m_c^2}{M^2(1-\alpha)} \right] \{ m_c^2 m_s \langle \bar{u}u \rangle^2 (1-\alpha) \\ & - 2M^2 m_s \langle \bar{u}u \rangle^2 (1-\alpha) + \alpha [m_c^3 - m_c M^2(1 \end{aligned}$$

$$\begin{aligned} & - \alpha) [\langle \bar{u}u \rangle^2 + \langle \bar{s}s \rangle^2] + \langle \bar{d}d \rangle^2 [-2M^2 m_s(1-\alpha) \\ & + m_c^3 \alpha - m_c M^2 \alpha(1-\alpha) + m_c^2 m_s(1-\alpha)] \} \\ & + \frac{m_c^2}{1728\pi^2 M^4} \exp \left[-\frac{m_c^2}{M^2} \right] (m_c m_s + M^2) \\ & \times [9 \langle \bar{u}g_s \sigma G u \rangle \langle \bar{d}g_s \sigma G d \rangle + 8\pi^2 \langle \alpha_s G^2/\pi \rangle \langle \bar{u}u \rangle \langle \bar{d}d \rangle]. \end{aligned} \quad (\text{A.17})$$

In expressions above, $\Theta(z)$ is Unit Step function, and

$$L = s\alpha(1-\alpha) - m_c^2 \alpha. \quad (\text{A.18})$$

Calculation of the correlation function $\Pi(M^2, s_0)$ implies integrations of the spectral density's components $\rho^{\text{pert.}}(s, \alpha)$ and $\rho^{\text{Dim3(4,5,6,7,8)}}(s, \alpha)$ over α and s . These functions contain $\sim 1/(1-\alpha)^n$ type factors, which at $\alpha = 1$ may lead to singularities and diverge integrals. These spectral densities, however, depend also on the function $\Theta[s\alpha(1-\alpha) - m_c^2 \alpha]$ that cuts off dangerous regions in α integration. The components of $\Pi(M^2)$ are functions of the Feynman parameter α , and have in denominators the factors $(1-\alpha)^n$ as well. But the function $\exp[-m_c^2/M^2(1-\alpha)]$ in these components effectively regulates possible singularities rendering finite relevant integrals.

References

1. R. Aaij et al. (LHCb), Phys. Rev. Lett. **125**, 242001 (2020)
2. R. Aaij et al. (LHCb), Phys. Rev. D **102**, 112003 (2020)
3. M. Karliner, J.L. Rosner, Phys. Rev. D **102**, 094016 (2020)
4. Z.G. Wang, Int. J. Mod. Phys. A **35**, 2050187 (2020)
5. H.X. Chen, W. Chen, R.R. Dong, N. Su, Chin. Phys. Lett. **37**, 101201 (2020)
6. S.S. Agaev, K. Azizi, H. Sundu, J. Phys. G **48**, 085012 (2021)
7. S.S. Agaev, K. Azizi, H. Sundu, Nucl. Phys. A **1011**, 122202 (2021)
8. J. He, D.Y. Chen, Chin. Phys. C **45**, 063102 (2021)
9. S.S. Agaev, K. Azizi, H. Sundu, Phys. Rev. D **106**, 014019 (2022)
10. X.G. He, W. Wang, R. Zhu, Eur. Phys. J. C **80**, 1026 (2020)
11. V.M. Abazov et al. (D0 Collaboration), Phys. Rev. Lett. **117**, 022003 (2016)
12. S.S. Agaev, K. Azizi, H. Sundu, Phys. Rev. D **93**, 094006 (2016)
13. W. Chen, H.X. Chen, X. Liu, T.G. Steele, S.L. Zhu, Phys. Rev. Lett. **117**, 022002 (2016)
14. S.S. Agaev, K. Azizi, H. Sundu, Eur. Phys. J. C **78**, 141 (2018)
15. S.S. Agaev, K. Azizi, H. Sundu, Phys. Lett. B **820**, 135630 (2021)
16. M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, Nucl. Phys. B **147**, 385 (1979)
17. M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, Nucl. Phys. B **147**, 448 (1979)
18. I.I. Balitsky, V.M. Braun, A.V. Kolesnichenko, Nucl. Phys. B **312**, 509 (1989)
19. V.M. Belyaev, V.M. Braun, A. Khodjamirian, R. Ruckl, Phys. Rev. D **51**, 6177 (1995)
20. B.L. Ioffe, A.V. Smilga, Nucl. Phys. B **232**, 109 (1984)
21. S.S. Agaev, K. Azizi, H. Sundu, Phys. Rev. D **93**, 074002 (2016)
22. Y. Kondo, O. Morimatsu, T. Nishikawa, Phys. Lett. B **611**, 93 (2005)
23. S.H. Lee, H. Kim, Y. Kwon, Phys. Lett. B **609**, 252 (2005)
24. W. Lucha, D. Melikhov, H. Sadjian, Phys. Rev. D **100**, 014010 (2019)
25. Y. Sarac, H. Kim, S.H. Lee, Phys. Rev. D **73**, 014009 (2006)

26. Z.G. Wang, Int. J. Mod. Phys. A **30**, 1550168 (2015)
27. S.S. Agaev, K. Azizi, B. Barsbay, H. Sundu, Nucl. Phys. B **939**, 130 (2019)
28. H. Sundu, S.S. Agaev, K. Azizi, Eur. Phys. J. C **79**, 215 (2019)
29. S.S. Agaev, K. Azizi, H. Sundu, Turk. J. Phys. **44**, 95 (2020)
30. B.L. Ioffe, Nucl. Phys. B **188**, 317 (1981) [Erratum: Nucl. Phys. B **191**, 591 (1981)]
31. B.L. Ioffe, Prog. Part. Nucl. Phys. **56**, 232 (2006)
32. S. Narison, Nucl. Part. Phys. Proc. **270–272**, 143 (2016)
33. P.A. Zyla et al. (Particle Data Group), Prog. Theor. Exp. Phys. **2020**, 083C01 (2020)
34. Z.G. Wang, Eur. Phys. J. C **75**, 427 (2015)