On the Numerical Analysis of a Nonlinear Fractional Schrödinger Equation with Neumann Boundary Condition

Betul Hicdurmaz[1,a)]

[1]*Department of Mathematics, Faculty of Engineering and Natural Sciences, Istanbul Medeniyet University, 34700, Istanbul, Turkey*

[a)]betul.hicdurmaz@medeniyet.edu.tr

Abstract. In the present paper, a nonlinear fractional Schrödinger equation is investigated with a numerical approach. First and second orders of accuracy difference schemes are constructed for a mixed problem for a nonlinear fractional Schrödinger equation with Neumann boundary condition. Results of numerical experiments support the convergence of solutions of constructed first and second order of accuracy difference schemes to exact solution of the problem.
Keywords: Fractional derivative, Convergence, Neumann boundary condition.
PACS: 02.30.Jr, 02.60.Lj, 02.70.Bf

INTRODUCTION

In this study, a mixed boundary value problem for the m-dimensional nonlinear fractional Schrödinger equation

$$\begin{cases} i\frac{\partial u}{\partial t} - \sum_{r=1}^{m}(a_r(x)u_{x_r})_{x_r} = \int\limits_0^t f\left(s, D_s^{\alpha}u(s,x)\right)ds, \\ 0 < t < T, x = (x_1, \cdots, x_m) \in \Omega, \\ u(0,x) = 0, \ x \in \overline{\Omega}, \\ \partial u/\partial \overrightarrow{n} = 0, \ x \in S \end{cases} \tag{1}$$

with a homogeneous Neumann condition is considered [1]. Here $a_r(x), \ x \in \Omega$ is a smooth function and $a_r(x) \geq a > 0$ for $1 \leq r \leq m$, a is a constant. $\overrightarrow{n}$ denotes the normal to the boundary $S = \partial\Omega$ and $\alpha \in [0,1]$.

Here, first and second orders of accuracy difference schemes will be presented for problem (1). Numerical experiments will be carried out on a mixed boundary value problem for a one-dimensional nonlinear fractional Schrödinger equation. Numerical results will be presented.

DISCRETIZATION OF PROBLEM

In order to discretize problem (1) with respect to space variable, we define the grid sets:

$$\begin{aligned} \overline{\Omega}_h = \{x_r = (h_1 r_1, ..., h_m r_m), r = (r_1, ..., r_m), \\ 0 \leq r_j \leq N_j, h_j N_j = 1, j = 1, ..., m\}, \\ \Omega_h = \overline{\Omega}_h \cap \Omega, S_h = \overline{\Omega}_h \cap S. \end{aligned} \tag{2}$$

Third International Conference of Mathematical Sciences (ICMS 2019)
AIP Conf. Proc. 2183, 070022-1–070022-4; https://doi.org/10.1063/1.5136184
Published by AIP Publishing. 978-0-7354-1930-8/$30.00

13 October 2025 06:57:28

In this step, we assign the spatial derivative operator in problem (1) with the difference operator A_h^x as:

$$A_h^x u^h = -\sum_{r=1}^{m} (a_r(x) u^h_{\overline{x_r}})_{x_r, j_r} \tag{3}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $D^h u^h(x) = 0$ for all $x \in S_h$. Here, $D^h u^h(x)$ represents an approximation of $\partial u/\partial \overrightarrow{n}$ [2]. It is beneficial to note that A_h^x is a self-adjoint positive definite operator in $L_2(\overline{\Omega}_h)$ [3]-[4]. With the help of A_h^x, we arrive at the initial value problem

$$\begin{cases} i\frac{\partial u^h(t,x)}{\partial t} + A_h^x u^h(t,x) = \int\limits_0^t f^h\left(s, D_s^\alpha u^h(s,x)\right) ds, \\ 0 < t < T, x \in \overline{\Omega}_h, \\ u(0,x) = 0, x \in \overline{\Omega}_h. \end{cases} \tag{4}$$

Next step is discretization over time. In order to achieve this we need first and second orders of accuracy difference formulas for fractional time derivative which are presented respectively as [5]:

$$D_\tau^{1,\alpha} u_k = \frac{1}{\Gamma(1-\alpha)} \sum_{m=1}^{k} \frac{\Gamma(k-m-\alpha+1)}{(k-m)!} \left(\frac{u_m - u_{m-1}}{\tau^\alpha}\right), \tag{5}$$

and

$$D_\tau^{2,\alpha} u_k = \begin{cases} \frac{1}{\Gamma(3-\alpha)2^{1-\alpha}\tau^\alpha}(-u_0 + u_1),\ k = 1, \\ \frac{3^{2-\alpha}}{2^{4-\alpha}\tau^\alpha \Gamma(4-\alpha)} \{(14\alpha - 15)u_0 - (\alpha + 33)u_1 \\ +(2\alpha + 3)u_2\},\ k = 2, \\ \sum_{m=2}^{k-1} \left\{ \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \eta(k-m)(u_{m-1} - u_{m-2}) \right. \\ + \frac{(u_m - 2u_{m-1} + u_{m-2})\tau^{-\alpha}}{\Gamma(1-\alpha)} \left(\frac{(k-m+1)\eta(k-m)}{(1-\alpha)}\right) \\ \left. + \frac{(u_m - 2u_{m-1} + u_{m-2})\tau^{-\alpha}}{\Gamma(1-\alpha)} \left(\frac{\zeta(k-m)}{(2-\alpha)}\right) \right\} \\ + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)2^{1-\alpha}}(u_{k-1} - u_{k-2}) \\ + \frac{\tau^{-\alpha}(3-\alpha)}{\Gamma(3-\alpha)2^{2-\alpha}}(u_k - 2u_{k-1} + u_{k-2}),\ 3 \le k \le N, \end{cases} \tag{6}$$

where

$$\eta(r) = (r + 1/2)^{1-\alpha} - (r - 1/2)^{1-\alpha}, \zeta(r) = (r - 1/2)^{2-\alpha} - (r + 1/2)^{2-\alpha}. \tag{7}$$

Using the approach on approximation of abstract parabolic equations in [4], we construct the first and second orders of accuracy difference schemes for problem (1) respectively as:

$$\begin{cases} i\frac{u_k^h - u_{k-1}^h}{\tau} + A_h^x u_k^h = \sum_{l=1}^{k} f_{l-1}^h(x)\tau,\ x \in \overline{\Omega}_h, \\ f_k^h(x) = f(t_k, x, D_\tau^{1,\alpha} u_k^h),\ t_k = k\tau,\ 1 \le k \le N-1,\ N\tau = 1, x \in \overline{\Omega}_h \\ u_0^h(x) = 0,\ x \in \overline{\Omega}_h \end{cases} \tag{8}$$

13 October 2025 06:57:28

and

$$\begin{cases} i\frac{u_k^h - u_{k-1}^h}{\tau} + \frac{1}{2}A_h^x u_k^h + \frac{1}{2}A_h^x u_{k-1}^h = F_k^h(x),\ x \in \overline{\Omega}_h, \\ t_k = k\tau,\ 1 \le k \le N-1,\ N\tau = 1, \\ u_0^h(x) = 0,\ x \in \overline{\Omega}_h \end{cases} \tag{9}$$

where

$$F_k^h(x) = \begin{cases} \frac{\tau}{2}\left(f(\tau, x, D_\tau^{2,\alpha}u_1^{(h)}) + f(0, x, D_\tau^{2,\alpha}u_0^{(h)})\right), k = 1, \\ +\frac{\tau}{2}\left(f(t_k, x, D_\tau^{2,\alpha}u_k^{(h)}) + f(t_{k-1}, x, D_\tau^{2,\alpha}u_{k-1}^{(h)})\right) \\ \tau\sum_{l=1}^{k-1} f(t_l, x, D_\tau^{2,\alpha}u_l^{(h)}), x \in \overline{\Omega}_h, 2 \le k \le N. \end{cases} \tag{10}$$

NUMERICAL ANALYSIS

We consider the following one-dimensional nonlinear fractional Schrödinger differential equation with the exact solution $u = (e^{it} - 1)\cos(x)$:

$$\begin{aligned} & i\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = \int_0^t \sin(D^\alpha(u(s,x) - (e^{it} - 1)\cos x))ds - \cos x \\ & 0 < x < \pi, 0 < t < 1, \\ & u(0,x) = 0, 0 < x < \pi, \\ & u_x(t,0) = u_x(t,\pi) = 0, 0 < t < 1. \end{aligned} \tag{11}$$

Here, we implement iterated first and second orders of difference schemes (8) and (9) on a one dimensional nonlinear fractional Schrödinger problem [6]. Fully discrete difference schemes are achieved by implementation of central difference method for discretization of spatial operator. Throughout the experiments, iterations start with $u_k^0 = 0$ and terminate when the maximum absolute error between each iteration becomes less than 10^{-7} at an iteration step. To get the first and second orders of accuracy difference schemes for problem (11), we implement difference schemes (8) and (9). Then, we convert the problem into a system of matrix equations and we use a modified Gauss-Elimination method to deal with this system. L_2 error-(E_1) and maximum error-(E_2) of first order difference scheme for problem (11) are reported in Table 1 and Table 2 for $\alpha = 0.5$ and $\alpha = 0.75$ respectively. Also, estimated rates of convergence are demonstrated in Table 1 and Table 2 for the time variable. A similar experiment is carried out for second order of accuracy difference scheme of problem (11) which is obtained by implementation of difference scheme (9) on problem (11). Numerical results are presented in Table 3 and Table 4 for $\alpha = 0.5$ and $\alpha = 0.75$ respectively. Here, it is useful to note that estimated rates of convergence are demonstrated in the tables for errors E_1 and E_2 as C_1 and C_2 respectively. Numerical results support the convergence of solutions of the constructed first and second orders of accuracy difference schemes to exact solution of problem (11).

TABLE 1. Errors of solution for first order difference scheme (8) for problem (11), where $\alpha = 0.50$ when $h = 0.002\pi$.

N	E_1	C_1	E_2	C_2
10	9.13×10^{-2}	0.95	1.05×10^{-1}	0.97
20	4.73×10^{-2}	0.93	5.36×10^{-2}	0.96
40	2.49×10^{-2}	0.87	2.76×10^{-2}	0.92
80	1.36×10^{-2}	-	1.46×10^{-2}	-

13 October 2025 06:57:28

TABLE 2. Errors of solution for first order difference scheme (8) for problem (11), where $\alpha = 0.75$ when $h = 0.002\pi$.

N	E_1	C_1	E_2	C_2
10	9.19×10^{-2}	0.95	1.06×10^{-1}	0.98
20	4.76×10^{-2}	0.92	5.39×10^{-2}	0.96
40	2.51×10^{-2}	0.85	2.78×10^{-2}	0.93
80	1.36×10^{-2}	-	1.46×10^{-2}	-

TABLE 3. Errors of solution for second order difference scheme (9) for problem (11), where $\alpha = 0.50$ when $h = 0.002\pi$.

N	E_1	C_1	E_2	C_2
10	9.03×10^{-4}	2.08	1.10×10^{-3}	2.04
20	2.12×10^{-4}	2.03	2.66×10^{-4}	2.04
40	5.16×10^{-5}	2.00	6.45×10^{-5}	2.02
80	1.29×10^{-5}	-	1.59×10^{-5}	-

TABLE 4. Errors of solution for second order difference scheme (9) for problem (11), where $\alpha = 0.75$ when $h = 0.002\pi$.

N	E_1	C_1	E_2	C_2
10	9.97×10^{-4}	2.16	1.30×10^{-3}	2.18
20	2.23×10^{-4}	2.07	2.79×10^{-4}	2.07
40	5.30×10^{-5}	2.00	6.64×10^{-5}	2.01
80	1.33×10^{-5}	-	1.64×10^{-5}	-

CONCLUSION

A nonlinear fractional Schrödinger equation is investigated with a numerical approach. The first and second order of accuracy difference schemes are constructed for a mixed type of nonlinear fractional Schrödinger problem with Neumann boundary condition. Results of numerical experiments support the convergence of solutions of the constructed difference schemes to exact solution of the problem.

ACKNOWLEDGMENTS

The author would like to thank to Prof. Allaberen Ashyralyev for his valuable comments that improved the manuscript.

REFERENCES

[1] A. Ashyralyev and B. Hicdurmaz, Mathematical Methods in Applied Sciences (to be published).
[2] C. Ashyralyyev and M. Dedeturk, Boundary Value Problems **51**, 1–15 (2015).
[3] S. G. Krein, *Linear Differential Equations in Banach Space* (Nauka, Moscow, Russia, 1966).
[4] A. Ashyralyev and P. E. Sobolevskii, *New Difference Schemes for Partial Differential Equations*, Operator Theory: Advances and Applications, Vol. 148 (Birkhauser, Basel, Boston, Berlin, 2004).
[5] A. Ashyralyev, N. Emirov, and Z. Cakir, Electronic Journal of Differential Equations **97**, 1–17 (2014).
[6] A. Ashyralyev and B. Hicdurmaz, AIP Conference Proceedings **1470**, 92–94 (2012).
[7] B. Hicdurmaz and A. Ashyralyev, Computers and Mathematics with Applications **72**, 1703–1713 (2016).

13 October 2025 06:57:28