J. Math. Anal. Appl. 460 (2018) 565–581

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On biconservative surfaces in 4-dimensional Euclidean space

Rüya Yeğin Şen [a,*], Nurettin Cenk Turgay [b]

[a] Istanbul Medeniyet University, Faculty of Engineering and Natural Sciences, Department of Mathematics, 34700 Üsküdar, Istanbul, Turkey
[b] Istanbul Technical University, Faculty of Science and Letters, Department of Mathematics, 34469 Maslak, Istanbul, Turkey

ARTICLE INFO

Article history:
Received 14 August 2017
Available online 8 December 2017
Submitted by J. Lenells

Keywords:
Biconservative surfaces
Parallel normalized mean curvature vectors
Meridian surfaces

ABSTRACT

In this paper, we study non-CMC biconservative surfaces with flat normal bundle in the Euclidean 4-space $\mathbb{E}^4$. First, we determine locally all parallel normalized mean curvature vector (PNMCV) biconservative surfaces in $\mathbb{E}^4$, showing that they are certain rotational surfaces. Then, we consider meridian surfaces lying on a rotational hypersurface of $\mathbb{E}^4$ and construct a family of biconservative surfaces with flat normal bundle and non-parallel normalized mean curvature vector.

1. Introduction

Let (M^m, g) and $(N^n, \tilde{g})$ be some Riemannian manifolds. Then, the bi-energy functional is defined by

$$E_2(\psi) = \frac{1}{2}\int_M |\tau(\psi)|^2 v_g$$

whenever $\psi : M \to N$ is a smooth mapping, where $\tau(\psi)$ denotes the tension field of ψ.

A mapping $\psi : M \to N$ is said to be biharmonic if it is a critical point of E_2. In [18], it was proved that the mapping ψ is biharmonic if and only if it satisfies the Euler–Lagrange equation associated with bi-energy functional given by

$$\tau_2(\psi) = 0, \tag{1}$$

where τ_2 is the bitension field defined by $\tau_2(\psi) = \Delta\tau(\psi) - \operatorname{tr}\tilde{R}(d\psi, \tau(\psi))d\psi$ and Δ is the Rough–Laplacian (see also [19]).

* Corresponding author.
E-mail addresses: ruya.yegin@medeniyet.edu.tr (R. Yeğin Şen), turgayn@itu.edu.tr (N.C. Turgay).

https://doi.org/10.1016/j.jmaa.2017.12.009
0022-247X/

Biharmonic isometric immersions have cough a particular interest of many geometers ([5,9,12,16,17,21,22]). When ψ is an isometric immersion, by considering tangential and normal components of $\tau_2(\psi)$, one can obtain the following proposition (see for example [9,22]).

Proposition 1.1. *Let $\psi : M^m \to N^n$ be an isometric immersion between two Riemannian manifolds. Then, ψ is biharmonic if and only if the equations*

$$m\mathrm{grad}\,\|H\|^2 + 4\mathrm{tr}A_{\nabla^\perp_\cdot H}(\cdot) + 4\mathrm{tr}(\tilde{R}(\cdot,H)\cdot)^T = 0 \tag{2}$$

and

$$\mathrm{tr}\,\alpha_\psi(A_H(\cdot),\cdot) - \Delta^\perp H + \mathrm{tr}\,(\tilde{R}(\cdot,H)\cdot)^\perp = 0 \tag{3}$$

are satisfied, where A, H and α_ψ denote the shape operator, the mean curvature vector and second fundamental form of ψ, $\nabla^\perp$ is the normal connection of M and $\Delta^\perp$ is the Laplacian associated with $\nabla^\perp$.

From Proposition 1.1, one can see that an isometric immersion $x : M \to N$ is biharmonic if its mean curvature vector field H vanishes identically. In [4], Bang-Yen Chen conjectured that the converse of this statement is also true if the ambient space is Euclidean. Chen's biharmonic conjecture has been verified in a lot of particular cases so far (see for example [5,6,8,12,16,21]). However, the conjecture is still open.

On the other hand, a mapping $\psi : M \to N$ satisfying the condition

$$\langle \tau_2(\psi), d\psi\rangle = 0, \tag{4}$$

that is weaker than (1), is said to be biconservative. In particular, if $\psi = \mathrm{x}$ is an isometric immersion, then (4) is equivalent to

$$\tau_2(\mathrm{x})^T = 0,$$

where $\tau_2(\mathrm{x})^T$ denotes the tangential part of $\tau_2(\mathrm{x})$. In this case, M is said to be a biconservative submanifold of N. Before we proceed, we would like to note that one can conclude the following well-known proposition, by considering Proposition 1.1 (see for example [1,9]).

Proposition 1.2. *Let $\mathrm{x} : M^m \to N^n$ be an isometric immersion between two Riemannian manifolds. Then, x is biconservative if and only if the equation (2) is satisfied.*

In order to understand geometry of biharmonic submanifolds, biconservative immersions have been studied in many papers so far. For example, the complete classification of biconservative hypersurfaces in Euclidean spaces with three distinct principle curvatures is obtained by the second named author in [25]. Later, this study moved into pseudo-Euclidean spaces in [14,26].

On the other hand, classification results on biconservative surfaces in 3-dimensional space-forms have been appeared in some papers. In [16], Hasanis and Vlachos proved that a biconservative surface with non-constant mean curvature in the Euclidean 3-space $\mathbb{E}^3$ is necessarily congruent to a rotational surface with an appropriately chosen profile curve. Later, Yu Fu obtained a classification result on biconservative surfaces in the Minkowski 3-space $\mathbb{E}^3_1$, [11]. Further, biconservative surfaces in the 3-dimensional Lorentzian space forms were studied in [13]. In [23], Montaldo et al. obtained the complete classification of biconservative surfaces with constant mean curvature in the 4-dimensional Riemannian space-forms. The complete classification of surfaces with parallel mean curvature vector in product spaces $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ appeared in [10]. Most recently, the general properties of biconservative surfaces in arbitrary manifolds, and in particular the properties of biconservative surfaces with constant mean curvature were studied in [24].

In this paper, we obtain new results on biconservative surfaces in the Euclidean 4-space. In Sect. 2, after we describe the notation that we will use, we give basic facts on biconservative submanifolds. In Sect. 3, we give complete classification of biconservative surfaces in Euclidean spaces with parallel normalized mean curvature vector field. Then, in Sect. 4, we get complete classification of biconservative meridian surfaces and construct a family of biconservative surfaces with flat normal bundle and non-parallel normalized mean curvature vector.

2. Biconservative submanifolds in Euclidean spaces

Let $\mathbb{E}^n$ denote the Euclidean n-space with the canonical positive definite Euclidean metric tensor given by

$$\widetilde{g} = \langle\ ,\ \rangle = \sum_{i=1}^{n} dx_i^2,$$

where $(x_1, x_2, \dots, x_n)$ is a rectangular coordinate system in $\mathbb{E}^n$ and $\tilde{\nabla}$ stands for its Levi-Civita connection. Consider an m-dimensional submanifold M of $\mathbb{E}^n$. Let $\psi : M \to \mathbb{E}^n$ be an isometric immersion and let $\nabla^\perp$ denote its normal connection. A normal vector field ξ is called parallel if $\nabla^\perp_X \xi = 0$ whenever X is tangent to M.

On the other hand, the mean curvature vector field H of ψ is defined by

$$H = \frac{1}{2}\mathrm{tr}h(\cdot,\cdot) \tag{5}$$

and the mean curvature (function) is given by

$$f = \langle H, H\rangle^{1/2},$$

where $h = \alpha_\psi$ is the second fundamental form of ψ. M is called minimal if f vanishes identically.

Assume that M is not minimal, i.e., $f > 0$. If the unit normal vector field H/f is parallel, then M is called a parallel normalized mean curvature vector field submanifold, i.e., M is a PNMCV submanifold. It is obvious that PNMCV submanifolds generalize non-minimal submanifolds with parallel mean curvature vector. Further, a PNMCV submanifold has parallel mean curvature vector if and only if it has constant mean curvature which is equivalent to being constant of f. It is possible to find examples of PNMCV submanifolds with non-constant mean curvature (see for example [3,20]). Before we proceed we would like to note that a PNMCV surface M in the Euclidean space $\mathbb{E}^n$ has flat normal bundle (see [2, Proposition 1.1, pg. 99]).

Since the curvature tensor $\tilde{R}$ of $\mathbb{E}^n$ vanishes identically, the following proposition is obtained immediately from (2).

Proposition 2.1. *Let M be an m-dimensional PNMCV submanifold of the Euclidean space $\mathbb{E}^n$ and $\psi : M \to \mathbb{E}^n$ an isometric immersion. Then, ψ is biconservative if and only if*

$$A_{m+1}(\mathrm{grad}f) = -\frac{mf}{2}\mathrm{grad}f, \tag{6}$$

where $e_{m+1} = \frac{H}{f}$ and A_{m+1} is the shape operator corresponding to e_{m+1}.

Remark 2.2. If the mean curvature vector H is parallel, then (6) is satisfied trivially. Therefore, after this point by a proper-PNMCV biconservative surface, we will mean a surface admitting a PNMCV biconservative immersion into $\mathbb{E}^4$ whose mean curvature and its gradient is nowhere vanishing.

2.1. Basic equations in the theory of surfaces of $\mathbb{E}^n$

Let M be a surface in $\mathbb{E}^n$, ∇, h and A denote its the Levi-Civita connection, second fundamental form and shape operator, respectively. Note that R and $\tilde{R}$ will stand for curvature tensor of M and $\mathbb{E}^n$, respectively.

For any tangent vector fields X, Y, Z on M the Codazzi equation $(\tilde{R}(X,Y)Z)^\perp = 0$ and the Gauss equation $(\tilde{R}(X,Y)Z)^T = 0$ take the form

$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z) \tag{7}$$

and

$$R(X,Y)Z = A_{h(Y,Z)}X - A_{h(X,Z)}Y, \tag{8}$$

respectively, where $(\bar{\nabla}_X h)(Y,Z)$ is defined by

$$(\bar{\nabla}_X h)(Y,Z) = \nabla^\perp_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

The Ricci equation $(\tilde{R}(X,Y)\xi)^\perp = 0$ takes the form

$$R^D(X,Y)\xi = h(X, A_\xi Y) - h(A_\xi X, Y) \tag{9}$$

whenever X, Y are tangent and ξ is normal to M.

3. Biconservative PNMCV surfaces in $\mathbb{E}^4$

In this section, we would like to obtain complete local classification of biconservative surfaces with parallel normalized mean curvature vector in the Euclidean 4-space $\mathbb{E}^4$.

First we would like to obtain shape operators and Levi-Civita connection of a biconservative PNMCV surface in $\mathbb{E}^4$.

For simplicity, we'll assume that all surfaces are oriented.

Lemma 3.1. *Let (M,g) be an abstract surface. Then, M admits a proper-PNMCV biconservative isometric immersion into $\mathbb{E}^4$ if and only if there exists an orthonormal frame field $\{e_1, e_2; e_3, e_4\}$ such that*

(1) *The Levi-Civita connection ∇ of M and normal connection $\nabla^\perp$ of ψ satisfy*

$$\nabla_{e_1} e_1 = \nabla_{e_1} e_2 = 0, \tag{10a}$$

$$\nabla_{e_2} e_1 = \frac{-3e_1(f)}{4f} e_2, \quad \nabla_{e_2} e_2 = \frac{3e_1(f)}{4f} e_1, \tag{10b}$$

$$\nabla^\perp e_3 = \nabla^\perp e_4 = 0. \tag{10c}$$

(2) *Shape operators along e_3 and e_4 have matrix representations given by*

$$A_3 = \begin{pmatrix} -f & 0 \\ 0 & 3f \end{pmatrix}, \qquad A_4 = \begin{pmatrix} cf^{3/2} & 0 \\ 0 & -cf^{3/2} \end{pmatrix} \tag{11}$$

for a non-negative constant c and a smooth non-vanishing function f such that $e_1(f) \neq 0$ and $e_2(f) = 0$.

Proof. Let the mean curvature vector of ψ be H and $\langle H, H\rangle = f^2$. In order to prove the necessary condition, we assume that ψ is biconservative and H is parallel. We choose the frame field $\{e_3, e_4\}$ of the normal bundle of M with $e_3 = \frac{H}{f}$. Note that (6) is satisfied for $m = 2$. On the other hand, by the assumption e_3 is parallel. e_4 is also parallel because the co-dimension of M is 2. Therefore, we have

$$\nabla^\perp_X e_3 = \nabla^\perp_X e_4 = 0$$

for any tangent vector field X which yields (10c). As H is proportional to e_3, we have

$$\frac{1}{2}\mathrm{tr}A_3 = f \quad \text{and} \quad \mathrm{tr}A_4 = 0. \tag{12}$$

Let $\{e_1, e_2\}$ be a frame field of the tangent bundle of M so that $A_3 = \mathrm{diag}(h^3_{11}, h^3_{22})$. Then, because of (6), we may assume $e_1 = \nabla f/\|\nabla f\|$ and $h^3_{11} = -f$. We will prove that the frame field $\{e_1, e_2; e_3, e_4\}$ satisfies the other conditions given in the lemma.

The first equation in (12) implies $h^3_{22} = 3f$. Since e_3 is parallel, the Ricci equation (9) for $X = e_1, Y = e_2$ and $\xi = e_3$ yields

$$4fh(e_1, e_2) = 0.$$

Therefore, we have $\langle A_4(e_1), e_2\rangle = 0$. This equation and the second equation in (12) imply

$$A_{e_3} = \begin{pmatrix} -f & 0 \\ 0 & 3f \end{pmatrix}, \quad \text{and} \quad A_{e_4} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$$

for a smooth function λ.

By using Codazzi equation (7) for $X = e_1$, $Y = Z = e_2$, we obtain

$$e_1(\lambda) = -2\lambda\omega_{12}(e_2) \quad \text{and} \quad e_1(f) = \frac{-4f}{3}\omega_{12}(e_2) \tag{13}$$

which imply

$$2fe_1(\lambda) = 3\lambda e_1(f). \tag{14}$$

The Codazzi equation (7) for $X = e_2$, $Y = Z = e_1$ gives

$$\omega_{12}(e_1) = 0, \quad e_2(\lambda) = 0. \tag{15}$$

By integrating equation (14) and considering (15), we have $\lambda = cf^{3/2}$ for some constant c. By replacing e_4 with $-e_4$ if necessary, we may assume $c \geq 0$. Moreover, since e_1 is proportional to ∇f, f satisfies $e_2(f) = 0$ and $e_1(f) \neq 0$. Therefore, we have the condition (1) of the lemma. On the other hand, the second equation in (13) and the first equation in (15) give the condition (2) of the lemma. Hence, we completed the proof of the necessary condition.

Converse immediately follows from (10c) and (11). $\square$

Remark 3.2. If the conditions given in Lemma 3.1 are satisfied for $c = 0$, then we have $A_4 = 0$ which implies $\widetilde{\nabla}_X e_4 = 0$ whenever X is tangent to M. Thus, e_4 is a constant vector which yields that M is a surface lying in $\mathbb{E}^3$. The study of biconservative and biharmonic surfaces in $\mathbb{E}^3$ are completed in [16] and [4,17], respectively.

By considering Lemma 3.1, we have the following non-existence result

Corollary 3.3. *There exists no biharmonic PNMCV surface in* $\mathbb{E}^4$.

Proof. Assume that there exists a biharmonic PNMCV surface in $\mathbb{E}^4$. Then, M is a biconservative surface satisfying (3). By considering condition (2) of Lemma 3.1, we compute the left-hand side of (3) to obtain

$$\begin{aligned} \operatorname{tr}\alpha_\psi(A_H(\cdot),\cdot) - \Delta^\perp H + \operatorname{tr}(\tilde{R}(\cdot,H)\cdot)^\perp =& (10f^3 e_3 - 4cf^{7/2}e_4) - (\Delta f e_3) + 0 \\ =& (10f^3 - \Delta f)e_3 - 4cf^{7/2}e_4. \end{aligned} \tag{16}$$

By combining (3) and (16) we obtain $c = 0$. Thus, Remark 3.2 implies that M is a biharmonic surface lying in $\mathbb{E}^3$. Therefore, M is minimal which yields a contradiction. □

As an immediate consequence of Lemma 3.1, we would like to state the following corollary.

Corollary 3.4. *If the surface* M *admits a biconservative PNMCV immersion into* $\mathbb{E}^4$, *then the Gaussian curvature* K *of* M *and the mean curvature* f *of* ψ *satisfy* $K = -3f^2 - c^2 f^3$ *for a constant* $c \geq 0$.

By considering Lemma 3.1 and Corollary 3.4, we obtain the following rigidity result.

Proposition 3.5. *If the surface* M *admits two proper biconservative PNMCV immersions into* $\mathbb{E}^4$, *then these immersions differ by an isometry of* $\mathbb{E}^4$.

Proof. Assume that $\psi_1 : M \to \mathbb{E}^4$ and $\psi_2 : M \to \mathbb{E}^4$ are proper biconservative PNMCV immersions. Let ${}^iT^\perp M$, ${}^iH$ and ${}^i\nabla^\perp$ denote the normal bundle, the mean curvature and the normal connection of ψ_i, respectively and $f_i = \langle {}^iH, {}^iH\rangle^{1/2}$, for $i = 1, 2$.

By Corollary 3.4, the Gaussian curvature K of M satisfies

$$-K = 3f_1^2 + c^2 f_1^3 = 3f_2^2 + \tilde{c}^2 f_2^3 \tag{17}$$

for some non-negative constants $c, \tilde{c}$. Thus, we have $XK = (-6f_1 - 3c^2 f_1^2)X(f_1) = (-6f_2 - 3\tilde{c}^2 f_1^2)X(f_2)$ whenever X is tangent to M which implies

$$\frac{\operatorname{grad} f_i}{\|\operatorname{grad} f_i\|} = -\frac{\operatorname{grad} K}{\|\operatorname{grad} K\|}, \ i = 1, 2.$$

Therefore, the tangent vector fields e_1, e_2 described in Lemma 3.1 do not depend on the biconservative immersion but only the surface M.

Let $\{e_1, e_2; {}^ie_3, {}^ie_4\}$ denote the frame field corresponding to ψ_i described in Lemma 3.1. Then we have $e_2(f_i) = 0$. Moreover, (10b) implies

$$\nabla_{e_2} e_1 = \frac{-3e_1(f_1)}{4f_1} e_2 = \frac{-3e_1(f_2)}{4f_2} e_2$$

which gives $f_1 = \zeta f_2$ for a function ζ satisfying $e_1(\zeta) = 0$, $\zeta > 0$. Since $e_2(f_i) = 0$, ζ is a constant. Furthermore, considering (17) implies $\zeta = 1$ and $c = \tilde{c}$. Thus we have $f_1 = f_2$. Therefore, the shape operators $A_{{}^ie_3}$ and $A_{{}^ie_4}$ satisfy

$$A_{{}^1e_3} = A_{{}^2e_3} \text{ and } A_{{}^1e_4} = A_{{}^2e_4}. \tag{18}$$

Now, consider the vector bundle isomorphism $\Psi : T_1^\perp M \to T_2^\perp M$ given by $\Psi({}^1H) = {}^2H$ and $\Psi({}^1e_4) = {}^2e_4$. Then, since $f_1 = f_2$, we have

$$\langle \Psi({}^1e_\alpha), \Psi({}^1e_\beta)\rangle = \langle {}^2e_\alpha, {}^2e_\beta\rangle,\ \alpha, \beta = 3, 4$$

and (18) implies

$$\Psi \circ \alpha_{\psi_1} = \alpha_{\psi_2}.$$

Moreover, since ψ_1 and ψ_2 are PNMCV, we have

$$\Psi \circ \left({}^1\nabla^\perp e_\alpha\right) = 0 = {}^2\nabla^\perp(\Psi({}^1e_\alpha)), \quad \alpha = 3, 4.$$

Hence, by the fundamental theorem of submanifolds, there exists an isometry $\tilde{\Psi}$ of $\mathbb{R}^4$ such that $\tilde{\Psi} \circ \psi_1 = \psi_2$ (see [7, Theorem 1.1 at pg. 7]). $\square$

Let M be a PNMCV surface in $\mathbb{E}^4$, f its mean curvature and $m \in M$ with $f(m) \neq 0$ and $(\operatorname{grad} f)(m) \neq 0$. Next, by using Lemma 3.1, we would like to construct a local coordinate system on a PNMCV biconservative surface M in $\mathbb{E}^4$ on a neighborhood of $m \in M$.

Lemma 3.6. *Let M be a proper-PNMCV biconservative surface. Then, around any point of M, there exists a local coordinate system (s, t) such that*

$$f = f(s), \qquad e_1 = \frac{\partial}{\partial s} \quad \textit{and} \quad e_2 = f(s)^{3/4}\frac{\partial}{\partial t}$$

Proof. Let $\{e_1, e_2; e_3, e_4\}$ be a local orthornormal frame field given in Lemma 3.1. Because of $\nabla_{e_1}e_2 = 0$, $\nabla_{e_2}e_1 = -\frac{3e_1(f)}{4f}e_2$, we have $[e_1, e_2] = \frac{3e_1(f)}{4f}e_2$ which gives $[e_1, Ee_2] = 0$ for any function E satisfying

$$e_1(E) = \frac{-3e_1(f)}{4f}E. \tag{19}$$

Thus, there exists a local coordinate system (s, t) such that $e_1 = \frac{\partial}{\partial s}$ and $Ee_2 = \frac{\partial}{\partial t}$.

Moreover, $e_2(f) = 0$ which yields $f = f(s)$ and because of (19) we can choose E as $E = f(s)^{-3/4}$. $\square$

Corollary 3.7. *Let M be a proper-PNMCV biconservative surface in $\mathbb{E}^4$. Then, the mean curvature of M satisfies the following ordinary differential equation*

$$\frac{9f'(s)^2}{16f(s)^{7/2}} + c^2 f(s)^{3/2} + 9f(s)^{1/2} = c_2^2 \tag{20}$$

for a positive constant c_2, where s is the local coordinate given in Lemma 3.6.

Proof. Consider a local orthonormal frame field $\{e_1, e_2; e_3, e_4\}$ given in Lemma 3.1 and local coordinate system (s, t) given in Lemma 3.6. Note that the Gauss equation (8) for $X = Z = \partial_s$ and $Y = \partial_t$ gives

$$f(s)f''(s) - \frac{7}{4}f'(s)^2 + 4f(s)^4 + \frac{4}{3}c^2 f(s)^5 = 0.$$

By multiplying this equation with $\frac{9f'(s)}{8f(s)^{9/2}}$ and integrating the equation obtained, we get (20) for a constant c_2 which can be assumed to be positive. $\square$

Lemma 3.8. *Let M be a proper biconservative PNMCV surface in $\mathbb{E}^4$, where f is the mean curvature of M in $\mathbb{E}^4$ and $e_1 = \frac{\nabla f}{\|\nabla f\|}$. Then,*

(a) An integral curve of e_1 lies on a 3-dimensional hyperplane of $\mathbb{E}^4$.
(b) The curvature and torsion of an integral curve of e_1 are

$$\kappa(s) = f(s)\sqrt{1 + c^2 f(s)}, \tag{21a}$$

$$\tau(s) = \frac{cf'(s)}{2\sqrt{f(s)}(1 + c^2 f(s))}. \tag{21b}$$

(c) Any two integral curves of e_1 are congruent.

Proof. Let $\{e_1, e_2; e_3, e_4\}$ be the local orthonormal frame on M given by Lemma 3.1 and we suppose that γ is an integral curve of e_1 and it is parametrized by $\gamma(s) = x(s, t_0)$. Let $T = \gamma'$ be tangent of M and consider the Frenet frame $\{T(s), N(s), B(s), \tilde{B}(s)\}$ of the curve γ on M with the Frenet formulas

$$\begin{aligned} \frac{dT}{ds} &= \kappa N, \\ \frac{dN}{ds} &= -\kappa T + \tau B, \\ \frac{dB}{ds} &= -\tau N + \tau_2 \tilde{B}, \\ \frac{d\tilde{B}}{ds} &= -\tau_2 B, \end{aligned} \tag{22}$$

where κ, τ and τ_2 are curvatures of γ.

We proceed to compute the curvature and torsion of an integral curve γ of e_1. By combining (11) with Gauss formula and considering Weingarten formula we obtain

$$\tilde{\nabla}_{T(s)} T(s) = -f(s)e_3(s) + cf(s)^{3/2} e_4(s), \tag{23}$$

$$\tilde{\nabla}_{e_1} e_3 = e_3'(s) = f(s)T \tag{24}$$

and

$$\tilde{\nabla}_{e_1} e_4 = e_4'(s) = -cf(s)^{3/2} T, \tag{25}$$

where $e_3(s), e_4(s)$ are restrictions of e_3 and e_4 to γ.

By combining (22) with (23), we get (21a) and

$$N(s) = \frac{1}{\kappa}\frac{dT}{ds} = \frac{-1}{\sqrt{1 + c^2 f(s)}} e_3(s) + c\sqrt{\frac{f(s)}{1 + c^2 f(s)}} e_4(s). \tag{26}$$

By differentiation of (26) with respect to e_1 and using (24), (25), we obtain

$$\frac{dN}{ds} = \frac{c^2 f'(s)}{2(1 + c^2 f(s))^{3/2}} e_3(s) + \frac{cf'(s)}{2\sqrt{f(s)}(1 + c^2 f(s))^{3/2}} e_4(s) - \kappa T. \tag{27}$$

By combining equations (21a), (22) and (27), we obtain the torsion τ of γ as given in (21b) and

$$B(s) = \frac{c\sqrt{f(s)}}{\sqrt{1+c^2 f(s)}} e_3(s) + \frac{1}{\sqrt{1+c^2 f(s)}} e_4(s). \tag{28}$$

By applying e_1 to the equation (28) and using (24), (25), we obtain

$$\frac{dB}{ds} = \tau N(s),$$

which yields that $\tau_2 = 0$. Hence, we have the part (a) and part (b) of the lemma.

Now, we want to show part (c) of the lemma. Let $m_1, m_2 \in M$ lie on the same integral curve of e_2. Consider the local coordinate system (s,t) given in Lemma 3.6. Note that $e_2(f) = 0$ and $\nabla_{e_1} e_2 = 0$ yields $e_2(e_1(f)) = 0$ which implies $e_1(f) = f'(s)$. Therefore, because of (21), any integral curves γ_1 and γ_2 of e_1 have the same curvature and torsion functions. Hence, they are congruent to each other. □

Now, we are ready to get main classification theorem.

Theorem 3.9. *Let M be a proper-PNMCV biconservative surface in $\mathbb{E}^4$. Then, locally, it is congruent to the rotational surface*

$$\mathrm{x}(s,t) = (\alpha_1(s)\cos t, \alpha_1(s)\sin t, \alpha_2(s), \alpha_3(s)) \tag{29}$$

with arc-length parametrized smooth profile curve $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$,

$$\alpha_1(s) = \frac{1}{c_2 f(s)^{3/4}}$$

whose curvature and torsion are given by (21).

Proof. We consider a local orthonormal frame field $\{e_1, e_2; e_3, e_4\}$ given in Lemma 3.1 with

$$e_1 = \frac{\partial}{\partial s} \quad \text{and} \quad e_2 = f(s)^{3/4}\frac{\partial}{\partial t},$$

where (s,t) is local coordinate system give in Lemma 3.6. Note that we also have $f = f(s)$ which satisfies (20) for a constant c_2 because of Corollary 3.7. Moreover, the induced metric of M is

$$g = ds \otimes ds + \frac{1}{f(s)^{3/2}} dt \otimes dt. \tag{30}$$

By combining (10b) with (11), we have

$$\tilde{\nabla}_{\partial_t}\partial_s = -\frac{3f'}{4f}\partial_t, \tag{31a}$$

$$\tilde{\nabla}_{e_2} e_2 = \frac{3f'}{4f}\partial_s + 3f e_3 - c f^{3/2} e_4, \tag{31b}$$

$$\tilde{\nabla}_{\partial_t} e_3 = -3f\partial_t, \tag{31c}$$

$$\tilde{\nabla}_{\partial_t} e_4 = c f^{3/2}\partial_t. \tag{31d}$$

Let $\mathrm{x} : M \to \mathbb{E}^4$ be an isometric immersion. Then, (31a) becomes

$$\mathrm{x}_{ts} = \frac{-3f'}{4f}\mathrm{x}_t.$$

By solving this equation, we get

$$x(s,t) = f(s)^{-3/4}\Theta(t) + \Gamma(s) \tag{32}$$

for some $\mathbb{E}^4$-valued smooth functions Θ, Γ.

By combining (32) with (31b), we obtain

$$f(s)^{3/4}\Theta''(t) + \frac{9f'(s)^2}{16f(s)^{11/4}}\Theta(t) - \frac{3f'(s)}{4f(s)}\Gamma'(s) - 3f(s)e_3 + cf(s)^{3/2}e_4 = 0.$$

By applying ∂_t to this equation and using (31a), (31c) and (31d) we obtain

$$\Theta'''(t) + \left(\frac{9f'(s)^2}{16f(s)^{7/2}} + c^2 f(s)^{3/2} + 9f(s)^{1/2}\right)\Theta'(t) = 0.$$

By combining this equation with (20), we get

$$\Theta'''(t) + c_2^2\Theta'(t) = 0.$$

Thus, Θ has the form $\Theta(t) = \cos(c_2 t)A_1 + \sin(c_2 t)A_2 + A_3$ for some constant vectors A_1, A_2, A_3. Therefore, (32) becomes

$$x(s,t) = f(s)^{-3/4}\cos(c_2 t)A_1 + f(s)^{-3/4}\sin(c_2 t)A_2 + f(s)^{-3/4}A_3 + \Gamma(s). \tag{33}$$

By considering (30), we obtain

$$\begin{aligned}
&\langle A_3, A_3\rangle = a_3^2, \qquad \langle A_2, A_2\rangle = \langle A_1, A_1\rangle = \frac{1}{c_2^2},\\
&\langle A_i, A_j\rangle = \langle \Gamma'(s), A_2\rangle = \langle \Gamma'(s), A_1\rangle = 0, \qquad \text{if } i \neq j
\end{aligned}$$

and

$$\langle \Gamma'(s), \Gamma'(s)\rangle - \frac{3f'(s)}{2f(s)^{7/4}}\langle A_3, \Gamma'(s)\rangle + \frac{9f'(s)^2}{16f(s)^{7/2}}\left(a_3^2 + \frac{1}{c_2^2}\right) = 1$$

for a non-zero constant a_1. Therefore, up to a suitable isometry of $\mathbb{E}^4$, we may assume

$$\begin{aligned}
A_3 &= (0, 0, a_1, 0),\\
A_2 &= \left(0, \frac{1}{c_2}, 0, 0\right),\\
A_1 &= \left(\frac{1}{c_2}, 0, 0, 0\right),\\
\Gamma(s) &= \left(0, 0, \alpha_2(s) - \frac{a_1}{f(s)^{3/4}}, \alpha_3(s)\right)
\end{aligned} \tag{34}$$

for some smooth functions α_2, α_3. Hence, (33) gives (29) after re-defining t properly. Since $t = \text{const.}$ is an integral curve of e_1, Lemma 3.8 implies that the curvature and torsion of the curve

$$\alpha(s) = \left(\frac{1}{c_2 f(s)^{3/4}}, \alpha_2(s), \alpha_3(s)\right) \tag{35}$$

are the functions κ and τ given in (21). $\square$

In order to determine all curves satisfying the properties given in Theorem 3.9, we obtain the following proposition.

Proposition 3.10. *Let $f:(a,b)\to\mathbb{R}$ be a positive smooth function with non-vanishing derivative and assume that it satisfies (20) for a positive constant c_2. Then, an arc-length parametrized curve $\alpha=\left(\frac{1}{c_2 f^{3/4}},\alpha_2,\alpha_3\right)$ with curvature κ (21a) has the form*

$$\alpha(s)=\left(\frac{1}{c_2 f(s)^{3/4}},\frac{1}{c_2}\int_{s_0}^{s}(c^2 f(\xi)^{3/2}+9f(\xi)^{1/2})^{1/2}\cos(\theta(\xi))d\xi\ ,\right.$$
$$\left.\frac{1}{c_2}\int_{s_0}^{s}(c^2 f(\xi)^{3/2}+9f(\xi)^{1/2})^{1/2}\sin(\theta(\xi))d\xi\right), \tag{36}$$

where θ is the function given by

$$\theta'(s)=\frac{2cc_2 f(s)^{5/4}}{c^2 f(s)+9}. \tag{37}$$

Conversely, the curve given by (36) has curvature κ and torsion τ given by (21).

Proof. Let f be a positive function satisfying (20) which is equivalent to

$$f'(s)=\frac{4}{3}\varepsilon\sqrt{-c^2 f(s)^5+c_2^2 f(s)^{7/2}-9f(s)^4} \tag{38}$$

for a constant $\varepsilon\in\{-1,1\}$. Since α is parametrized by its arc-length, we have $\alpha_1'(s)^2+\alpha_2'(s)^2+\alpha_3'(s)^2=1$, where $\alpha_1=\frac{1}{c_2 f^{3/4}}$. By a direct computation considering this equation and (38), we get the condition

$$\alpha_2'(s)^2+\alpha_3'(s)^2=\frac{\left(c^2 f(s)^{3/2}+9f(s)^{1/2}\right)}{c_2^2}.$$

Therefore, we have (36) for a smooth function θ. Next, we use the condition $\kappa(s)=\langle T'(s),T'(s)\rangle^{1/2}$ to obtain (37). The remaining part of the proof follows from a further direct computation. □

Next, we obtain the converse of the above theorem.

Theorem 3.11. *Let M be the simple rotational surface in $\mathbb{E}^4$ given by (29) with arc-length parametrized smooth profile curve described in Proposition 3.10. Then, M is a PNMCV biconservative surface. Furthermore, its mean curvature is f.*

Proof. Consider the curve $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ with $\alpha_1=\frac{1}{c_2 f(s)^{3/4}}$ described in Proposition 3.10. Recall that its curvature κ and torsion τ given by (21). Let $t=(t_1,t_2,t_3)$, $n=(n_1,n_2,n_3)$, $b=(b_1,b_2,b_3)$ be the unit tangent, unit normal and unit binormal of α. Note that we have

$$t_1=\alpha_1'=-\frac{\varepsilon\sqrt{-c^2 f(s)^5+c_2^2 f(s)^{7/2}-9f(s)^4}}{c_2 f(s)^{7/4}}, \tag{39a}$$

$$n_1=\frac{t_1'}{\kappa}=\frac{f(s)^{1/4}\left(c^2 f(s)+3\right)}{c_2\sqrt{c^2 f(s)+1}}, \tag{39b}$$

$$b_1=-\frac{2cf(s)^{3/4}}{c_2\sqrt{c^2 f(s)+1}}. \tag{39c}$$

Now, let M be the simple rotational surface in $\mathbb{E}^4$ given by (29) with profile curve α. Consider the local orthonormal frame field $\{e_1, e_2, \hat{e}_3, \hat{e}_4\}$ on M given by

$$\begin{aligned} e_1 &= \frac{\partial}{\partial s}, \quad e_2 = \frac{1}{\alpha(s)}\frac{\partial}{\partial t}, \\ \hat{e}_3 &= (n_1(s)\cos t, n_1(s)\sin t, n_2(s), n_3(s)), \\ \hat{e}_4 &= (b_1(s)\cos t, b_1(s)\sin t, b_2(s), b_3(s)). \end{aligned}$$

By a direct computation considering (39), one can obtain (10a), (10b) and

$$A_{\hat{e}_3} = \begin{pmatrix} f(s)\sqrt{c^2 f(s)+1} & 0 \\ 0 & -\frac{f(s)(c^2 f(s)+3)}{\sqrt{c^2 f(s)+1}} \end{pmatrix}, \qquad A_{\hat{e}_4} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{2cf(s)^{3/2}}{\sqrt{c^2 f(s)+1}} \end{pmatrix}. \tag{40}$$

Furthermore, the normal connection of M satisfies

$$\begin{aligned} \nabla^{\perp}_{e_2}\hat{e}_3 &= 0, \\ \nabla^{\perp}_{e_1}\hat{e}_3 &= \frac{2c\varepsilon\sqrt{c_2^2 f(s)^{5/2} - f(s)^3\left(c^2 f(s)+9\right)}}{3c^2 f(s)+3}\hat{e}_4. \end{aligned} \tag{41}$$

Note that the mean curvature vector of M is

$$H = -\frac{f(s)}{\sqrt{c^2 f(s)+1}}\hat{e}_3 + \frac{cf(s)^{3/2}}{\sqrt{c^2 f(s)+1}}\hat{e}_4$$

which yields that the mean curvature of M is $H = f$. Thus, the normalized mean curvature vector of M is

$$e_3 = -\frac{1}{\sqrt{c^2 f(s)+1}}\hat{e}_3 + \frac{cf(s)^{1/2}}{\sqrt{c^2 f(s)+1}}\hat{e}_4 \tag{42}$$

and we put

$$e_4 = \frac{cf(s)^{1/2}}{\sqrt{c^2 f(s)+1}}\hat{e}_3 + \frac{1}{\sqrt{c^2 f(s)+1}}\hat{e}_4 \tag{43}$$

(40), (42) and (43) give (11). Furthermore, (10c) follows from a direct computation considering (38), (41) and (42). Thus, M is a PNMCV surface and the orthonormal frame field $\{e_1, e_2; e_3, e_4\}$ satisfies conditions of Lemma 3.1 which yields that M is biconservative. □

Now let (M, g) be the biconservative rotational surface in given by (29) in $\mathbb{E}^4$ with arc-length parametrized smooth profile curve described in Proposition 3.10 for a function θ satisfying (37). Define

$$\begin{aligned} \zeta_1(f) &= \frac{3}{4c_2}\int_{f_0}^{f}\left(\frac{c^2\xi^{3/2}+9\xi^{1/2}}{-c^2\xi^5 + c_2^2\xi^{7/2} - 9\xi^4}\right)^{1/2}\cos\tilde{\theta}(\xi)d\xi, \\ \zeta_2(f) &= \frac{3}{4c_2}\int_{f_0}^{f}\left(\frac{c^2\xi^{3/2}+9\xi^{1/2}}{-c^2\xi^5 + c_2^2\xi^{7/2} - 9\xi^4}\right)^{1/2}\sin\tilde{\theta}(\xi)d\xi, \\ \tilde{\theta}(f) &= \int_{f_0}^{f}\frac{3cc_2\xi^{5/4}}{2(c^2\xi+9)\sqrt{-c^2\xi^5 + c_2^2\xi^{7/2} - 9\xi^4}}d\xi, \end{aligned}$$

where $f_0 = f(s_0)$. Then, a parametrization of M is

$$\tilde{x}(f,t) = \frac{1}{c_2}\left(\frac{1}{f^{3/4}}\cos t, \frac{1}{f^{3/4}}\sin t, \zeta_1(f), \zeta_2(f)\right), \tag{44}$$

where $\tilde{x}$ is defined on an open subset of $(0,\infty)\times\mathbb{R}$ to be explained. Note that for the parametrization given by (44), we have

$$\langle \tilde{x}_f, \tilde{x}_f\rangle = E_{c,c_2}(f), \quad \langle \tilde{x}_f, \tilde{x}_t\rangle = 0, \quad \langle \tilde{x}_t, \tilde{x}_t\rangle = G(f),$$

where

$$E_{c,c_2}(f) = \frac{9}{16(-c^2f^5 + c_2^2 f^{7/2} - 9f^4)}, \tag{45}$$

$$G_{c_2}(f) = \frac{1}{c_2^2 f^{3/2}} \tag{46}$$

By a simple computation, we also see that $E_{c,c_2}(f) > 0$ if and only if $f \in (0, \delta_{c,c_2})$, where $\delta_{c,c_2} \in (0, c_2^{4/3}c^{-4/3})$ is the only positive root of

$$-c^2\delta^{3/2} + c_2^2 - 9\delta^{1/2} = 0. \tag{47}$$

Hence, we have

Proposition 3.12. *Let (M,g) be a surface. If M admits a biconservative PNMCV isometric immersion into $\mathbb{E}^4$, then it is isometric to*

$$\left(D_{c,c_2}, g_{c,c_2} = E_{c,c_2}(f)df^2 + G_{c_2}(f)dt^2\right) \tag{48}$$

for some positive constants c, c_2, where E_{c,c_2}, G are functions defined in (45), (46) and $D_{c,c_2} = (0,\delta_{c,c_2})\times\mathbb{R}$ for the positive root δ_{c,c_2} of (47).

By the next theorem, we prove that we get a 2-parameter family of proper-PNMCV biconservative surfaces in $\mathbb{E}^4$ by (29).

Theorem 3.13. *Let (D_{c,c_2}, g_{c,c_2}) and $(D_{\tilde{c},\tilde{c}_2}, g_{\tilde{c},\tilde{c}_2})$ denote the surfaces described in Proposition 3.12 for some non-negative constants $c, \tilde{c}$ and positive constants $c_2, \tilde{c}_2$. Then, the surfaces (D_{c,c_2}, g_{c,c_2}) and $(D_{\tilde{c},\tilde{c}_2}, g_{\tilde{c},\tilde{c}_2})$ are isometric if and only if $c = \tilde{c}$ and $c_2 = \tilde{c}_2$.*

Proof. In the proof this theorem, we use the notation $g = g_{c,c_2} = E_{c,c_2}(f)df^2 + G_{c_2}dt^2$, $\tilde{g} = g_{\tilde{c},\tilde{c}_2} = E_{\tilde{c},\tilde{c}_2}(\tilde{f})d\tilde{f}^2 + G_{\tilde{c}_2}(\tilde{f})d\tilde{t}^2$. We also put

$$\left.\frac{\partial}{\partial f}\right|_{(f,t)} = \partial_f, \ \left.\frac{\partial}{\partial t}\right|_{(f,t)} = \partial_t, \quad \left.\frac{\partial}{\partial \tilde{f}}\right|_{\Phi(f,t)} = \partial_{\tilde{f}}, \ \left.\frac{\partial}{\partial \tilde{t}}\right|_{\Phi(f,t)} = \partial_{\tilde{t}}$$

for a $(f,t) \in D_{c,c_2}$.

By a direct computation, we obtain the Gaussian curvature of (D_{c,c_2}, g) and $(D_{\tilde{c},\tilde{c}_2}, \tilde{g})$ as

$$K(f,t) = -f^2 - c^2 f^3 \quad \text{and} \quad \tilde{K}(\tilde{f},\tilde{t}) = -\tilde{f}^2 - \tilde{c}^2\tilde{f}^3, \tag{49}$$

respectively. In order to prove necessary condition, let $\Phi = (\Phi_1, \Phi_2) : (D_{c,c_2}, g) \to (D_{\tilde{c},\tilde{c}_2}, \tilde{g})$ be an isometry. Then we have $\tilde{K}(\Phi_1(f,t), \Phi_2(f,t)) = K(f,t)$ which is equivalent to

$$f^2 + c^2 f^3 = \Phi_1^2 + \tilde{c}^2 \Phi_1^3 \tag{50}$$

which implies $\Phi_1(f,t) = \Phi_1(f)$. Therefore, we have

$$d\Phi_{(f,t)}\left(\partial_f\right) = \Phi_1'(f)\partial_{\tilde{f}} + \frac{\partial \Phi_2(f,t)}{\partial f}\partial_{\tilde{t}}$$

and

$$d\Phi_{(f,t)}\left(\partial_t\right) = \frac{\partial \Phi_2(f,t)}{\partial t}\partial_{\tilde{t}},$$

where $d\Phi_{(f,t)} : T_{(f,t)}D_{c,c_2} \to T_{\Phi(f,t)}D_{\tilde{c},\tilde{c}_2}$ stands for the differential of Φ at (f,t).

From the equations $\tilde{g}_{\Phi(f,t)}\left(d\Phi_{(f,t)}\left(\partial_f\right), d\Phi_{(f,t)}\left(\partial_t\right)\right) = g_{(f,t)}\left(\partial_f, \partial_t\right) = 0$ and $\tilde{g}_{\Phi(f,t)}(d\Phi_{(f,t)}(\partial_t), d\Phi_{(f,t)}(\partial_t)) = g_{(f,t)}(\partial_t, \partial_t)$ we obtained

$$\frac{\partial \Phi_2(f,t)}{\partial f}\frac{\partial \Phi_2(f,t)}{\partial t} G_{\tilde{c}_2}(\Phi_1(f)) = 0, \tag{51}$$

$$\left(\frac{\partial \Phi_2(f,t)}{\partial t}\right)^2 G_{\tilde{c}_2}(\Phi_1(f)) = G_{c_2}(f). \tag{52}$$

Now, (52) yields $\frac{\partial \Phi_2(f,t)}{\partial t} \neq 0$. Thus, from (51) we get $\frac{\partial \Phi_2(f,t)}{\partial f} = 0$ from which we have $\Phi_2(f,t) = \Phi_2(t)$. Therefore, (52) implies

$$\left(\Phi_2'(t)\right)^2 = \frac{G_{c_2}(f)}{G_{\tilde{c}_2}(\Phi_1(f))} = \hat{b} \tag{53}$$

for a constant $\hat{b} \neq 0$. Therefore, we have

$$\phi_1(f) = bf \tag{54}$$

for a constant b which is positive because $c_2, \tilde{c}_2, f, \Phi_1(f) > 0$.

Now, by combining (50) and (54), we get

$$(1-b^2)f^2 + (c^2 - \tilde{c}^2 b^3)f^3 = 0$$

from which we obtain $\Phi_1(f) = f$ and $c = \tilde{c}$. By a further computation considering $\tilde{g}_{\Phi(f,t)}(d\Phi_{(f,t)}(\partial_f), d\Phi_{(f,t)}(\partial_f)) = g_{(f,t)}(\partial_f, \partial_f)$, we also obtain $c_2 = \tilde{c}_2$. Hence, the necessary condition is proved. The converse is obvious. □

4. Biconservative meridian surfaces in $\mathbb{E}^4$

In this section we study meridian surfaces in the Euclidean space $\mathbb{E}^4$. We get complete classification of biconservative meridian surfaces and construct biconservative surfaces with non-parallel normalize mean curvature vector.

Let $\alpha : I \to \mathbb{S}^2(1)$ be an arc-length parametrized smooth curve with (spherical) curvature κ and unit normal n, $i : \mathbb{S}^2(1) \to \mathbb{E}^4$ canonical inclusion and

$$(i \circ \alpha)(u) = (\alpha_1(u), \alpha_2(u), \alpha_3(u)),$$

where I is an open interval. Note that the unit tangent vector t and n satisfies

$$\alpha' = t, \tag{55}$$

$$t' = \kappa n - \alpha, \tag{56}$$

$$n' = -\kappa t. \tag{57}$$

Consider the planar curve $(f(v), v), v \in J$ for a smooth function f, where J is an open interval. Then the surface M in $\mathbb{E}^4$ given by

$$\mathrm{x}(u, v) = (f(v)\alpha_1(u), f(v)\alpha_2(u), f(v)\alpha_3(u), v), u \in I, v \in J \tag{58}$$

is said to be a meridian surface lies on a rotational hypersurface M^3 in $\mathbb{E}^4$ obtained by α around the x_4-axis in $\mathbb{E}^4$ [15].

Remark 4.1. It is well-know that the curve α becomes a circle with radius less than 1 if its (spherical) curvature κ is a non-zero constant. In this case, M becomes a rotational surface with profile curve lying in a 2-plane of $\mathbb{E}^4$. More precisely, in this case, M is congruent to the surface given by

$$\mathrm{x}(u, v) = (af(v)\cos u, af(v)\sin u, bf(v), v) \tag{59}$$

for some positive constants a, b with $a^2 + b^2 = 1$. Moreover, if $\kappa \equiv 0$ on I, then M is congruent to (59) with $a = 1$, $b = 0$. In this case, M is a rotational surface in $\mathbb{E}^3$. In [16], it is proved that the only biconservative surfaces in $\mathbb{E}^3$ are rotational ones. Furthermore, the profile curve of a biconservative rotational surface in $\mathbb{E}^3$ is obtained. Therefore, after this point we assume that κ does not vanish on I.

The tangent space of M given by (58) is spanned by the vector fields

$$e_1 = \frac{1}{E^{1/2}}\partial_v, \qquad e_2 = \frac{1}{f(v)}\partial_u, \tag{60}$$

where we put

$$E = E(v) = 1 + f'(v)^2.$$

Moreover, the normal vector fields

$$\begin{aligned} e_3 =& \frac{1}{E^{1/2}}\left(-\alpha_1(u), -\alpha_2(u), -\alpha_3(u), f'(v)\right), \\ e_4 =& (n(u), 0) \end{aligned} \tag{61}$$

form an orthonormal frame field for the normal bundle of M.

By a direct computation, we observe that e_3 and e_4 are parallel. Furthermore, the matrix representation of the shape operators of M are

$$A_{e_3} = \begin{pmatrix} -\frac{f''(v)}{E^{3/2}} & 0 \\ 0 & \frac{1}{E^{1/2}f(v)} \end{pmatrix}, \qquad A_{e_4} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\kappa(u)}{f(v)} \end{pmatrix}. \tag{62}$$

Thus, the mean curvature vector field H is

$$H = \frac{-f(v)f''(v) + E}{2f(v)E^{3/2}} e_3 + \frac{\kappa(u)}{2f(v)} e_4. \tag{63}$$

By considering (63), we obtain

$$\begin{aligned} 2 \operatorname{grad} \|H\|^2 =& E^{-9/2} f(v)^{-3} \Big(- E^3 f'(v) \left(\kappa(u)^2 E + 1\right) - 3f(v)^3 f'(v) f''(v)^3 \\ &+ E f(v)^2 \left(4f'(v)f''(v)^2 + f'''(v) \left(f(v)f''(v) - E\right)\right) \Big) e_1 \\ &+ \frac{\kappa(u)\kappa'(u)}{f(v)^3} e_2 \end{aligned} \tag{64}$$

and

$$\begin{aligned} \operatorname{tr} A_{\nabla^\perp_{\cdot} H}(\cdot) =& \frac{E^{-9/2} f(v)^{-2} f''(v)}{2} \Big(E f(v)^2 f'''(v) + f'(v) \left(E^2 - 3f(v)^2 f''(v)^2\right. \\ &\left. + E f(v) f''(v)\right) \Big) e_1 + \frac{\kappa(u)\kappa'(u)}{2f(v)^3} e_2. \end{aligned} \tag{65}$$

Now, assume that M is a biconservative meridian surface M lying in $\mathbb{E}^4$. Then, (2) is satisfied for $m = 2$ and we have $\tilde{R} = 0$ because the ambient space $\mathbb{E}^4$. By combining (2) with (64) and (65), we obtain

$$\begin{aligned} &-E^3 f'(v) \left(E\kappa(u)^2 + 1\right) + 2E^2 f(v) f'(v) f''(v) - 9f(v)^3 f'(v) f''(v)^3 \\ &+ 6E f(v)^2 f'(v) f''(v)^2 - E f(v)^2 f'''(v) \left(E - 3f(v)f''(v)\right) = 0 \end{aligned}$$

and

$$\kappa(u)\kappa'(u) = 0. \tag{66}$$

From (66) we see that κ is constant. Therefore, Remark 4.1 implies that M is a rotational surface. By considering $\kappa(u) = \kappa_0$, we obtain the following theorem.

Theorem 4.2. *A meridian surface M in $\mathbb{E}^4$ is biconservative if and only if it is congruent to the rotational surface given by*

$$\mathrm{x}(u, v) = \left(af(v)\cos u, af(v)\sin u, bf(v), v\right) \tag{67}$$

for some positive constants a, b with $a^2 + b^2 = 1$ and a smooth function f satisfying

$$\begin{aligned} &f'(v)(f'(v)^2 + 1)^3 \left(\kappa_0^2 \left(f'(v)^2 + 1\right) + 1\right) - 2f(v)f'(v) \left(f'(v)^2 + 1\right)^2 f''(v) \\ &+ 9f(v)^3 f'(v) f''(v)^3 - 6f(v)^2 f'(v) \left(f'(v)^2 + 1\right) f''(v)^2 \\ &+ f(v)^2 f'''(v) \left(f'(v)^2 + 1\right) \left(-3f(v)f''(v) + f'(v)^2 + 1\right) = 0, \end{aligned} \tag{68}$$

where $\kappa_0 = b/a$.

Acknowledgments

This work was obtained during the project supported by Research Fund of the Istanbul Medeniyet University (Project Number: F-GAP-2017-986).

We would like to express our sincere gratitude to the anonymous referee for his/her helpful comments that will help to improve the quality of the manuscript.

References

[1] R. Caddeo, S. Montaldo, C. Oniciuc, P. Piu, Surfaces in the three-dimensional space forms with divergence-free stress-bienergy tensor, Ann. Mat. Pura Appl. 193 (2014) 529–550.
[2] B.-Y. Chen, Geometry of Submanifolds, Marcel Dekker Inc., New York, 1973.
[3] B.-Y. Chen, Surfaces with parallel normalized mean curvature vector, Monatsh. Math. 90 (1980) 185–194.
[4] B.-Y. Chen, Some open problems and conjectures on submanifolds of finite type, Soochow J. Math. 17 (2) (1991) 169–188.
[5] B.-Y. Chen, A report on submanifolds of finite type, Soochow J. Math. 22 (1996) 117–337.
[6] B.-Y. Chen, M.I. Munteanu, Biharmonic ideal hypersurfaces in Euclidean spaces, Differential Geom. Appl. 31 (1) (2013) 1–16.
[7] M. Dajczer, Submanifolds and Isometric Immersions, Publish or Perish, Houston, 1990.
[8] I. Dimitric, Submanifolds of $\mathbb{E}^m$ with harmonic mean curvature vector, Bull. Inst. Math. Acad. Sin. 20 (1992) 53–65.
[9] D. Fetcu, S. Nistor, C. Oniciuc, On biconservative surfaces in 3-dimensional space forms, Comm. Anal. Geom. 24 (2016) 1027–1045.
[10] D. Fetcu, C. Oniciuc, A.L. Pinheiro, CMC biconservative surfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$, J. Math. Anal. Appl. 425 (2015) 588–609.
[11] Y. Fu, On bi-conservative surfaces in Minkowski 3-space, J. Geom. Phys. 66 (2013) 71–79.
[12] Y. Fu, Biharmonic hypersurfaces with three distinct principal curvatures in Euclidean 5-space, J. Geom. Phys. 75 (2014) 113–119.
[13] Y. Fu, Explicit classification of biconservative surfaces in Lorentz 3-space forms, Ann. Mat. Pura Appl. 194 (4) (2015) 805–822.
[14] Y. Fu, N.C. Turgay, Complete classification of biconservative hypersurfaces with diagonalizable shape operator in Minkowski 4-space, Int. J. Math. 27 (2016) 1650041.
[15] G. Ganchev, V. Milousheva, Invariants and Bonnet-type theorem for surfaces in $\mathbb{R}^4$, Cent. Eur. J. Math. 8 (2010) 993–1008.
[16] T. Hasanis, I. Vlachos, Hypersurfaces in $\mathbb{E}^4$ with harmonic mean curvature vector field, Math. Nachr. 172 (1995) 145–169.
[17] G.Y. Jiang, 2-harmonic isometric immersions between Riemannian manifolds, Chinese Ann. Math. Ser. A 7 (1986) 130–144.
[18] G.Y. Jiang, 2-harmonic maps and their first and second variational formulas, Chinese Ann. Math. Ser. A 7 (1986) 389–402.
[19] G.Y. Jiang, The conservation law for 2-harmonic maps between Riemannian manifolds, Acta Math. Sin. 30 (1987) 220–225.
[20] D.S.P. Leung, The Cauchy problem for surfaces with parallel normalized mean curvature vector in a four dimensional Riemannian manifold, preprint.
[21] S. Maeta, Biminimal properly immersed submanifolds in the Euclidean spaces, J. Geom. Phys. 62 (2012) 2288–2293.
[22] S. Montaldo, C. Oniciuc, A. Ratto, Proper biconservative immersions into the Euclidean space, Ann. Mat. Pura Appl. 195 (2016) 403–422.
[23] S. Montaldo, C. Oniciuc, A. Ratto, Biconservative surfaces, J. Geom. Anal. 26 (2016) 313–329.
[24] S. Nistor, On biconservative surfaces, Differential Geom. Appl. 54 (2017) 490–502.
[25] N.C. Turgay, H-hypersurfaces with 3 distinct principal curvatures in the Euclidean spaces, Ann. Mat. Pura Appl. 194 (2015) 1795–1807.
[26] A. Upadhyay, N.C. Turgay, A classification of biconservative hypersurfaces in a pseudo-Euclidean space, J. Math. Anal. Appl. 444 (2) (2016) 1703–1720.