Partial Differential Equations in Applied Mathematics 11 (2024) 100785

Contents lists available at ScienceDirect

Partial Differential Equations in Applied Mathematics

journal homepage: www.elsevier.com/locate/padiff

Analytical solutions and physical interpretation of a predator–prey system with Allee effect using fractional derivative operators

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ARTICLE INFO

Keywords:
Predator–prey system
Exact solution
Fractional derivative operators
Generalized exponential function method
Bäcklund transformation approach

ABSTRACT

This study investigates the dynamics of a predator–prey system by incorporating the Allee effect. The exact analytical solutions of the system are explored using the generalized exponential function method and the Bäcklund transformation approach. The system is discussed using Conformable, M-truncated, and Beta time fractional derivative operators to provide a physical interpretation. The effects of these derivative operators on the solutions are demonstrated through various 2D and 3D graphical simulations. Furthermore, the impacts of physical parameters on the model are analysed, providing valuable insights into the dynamics of Predator–prey systems with the Allee effect. Our obtained solutions differ from those in the literature, owing to the methodologies employed and the choice of differential operators utilized. Nevertheless, we believe that the comparisons presented for the differential operators are effective and beneficial.

1. Introduction

Partial differential equations of the nonlinear evolution type play an important role in modelling many phenomena occurring in nature. Nonlinear evolution type partial differential equations (NEPDEs) are used in modelling phenomena that occur in numerous fields such as physics, quantum, chemistry, ecology, biomathematics, biology and economics. However, obtaining exact and numerical solutions of NEPDEs is the widespread study of applied mathematics.[1–7] Many methods have been developed and continue to be developed for the formation of exact solutions is being studied. Fractional differential equations (FDEs) have emerged as a significant mathematical tool in the last several years for the analysis of several phenomena in science and engineering.

FDEs are used to simulate many different physical processes, including electricity, fluid dynamics, quantum physics, ecological systems, and more. There are several specified fractional derivative operators, and the derivatives about a space or time variable are fractional. By rephrasing the models in terms of the order derivative, they were improved. Furthermore, a significant and well-liked area of research is the exact solution techniques employed for partial differential equations for FDEs.

Scientists have investigated population dynamics using mathematical modelling. Beginning with the seminal works of Lotka (1926),[8] Volterra (1927),[9] and Fisher (1937),[10] this subject has promoted a broad range of scientific exchange between ecology and mathematics. Predicting ecological systems requires an understanding of the connection between predator and prey. The interaction between biological species in general and predator–prey species in particular has received a lot of attention lately, with an emphasis on proving the survival (or non-survival) of the creatures being studied.[11] To predict these interactions and the survival of different species several effective biological mathematical models have been proposed.[12–15] The analysis of spatial distribution patterns and mechanisms of interacting species is a current research topic in conservation biology, mathematical ecology and biochemical reactions. Different types of living organisms compete or consume limited resources. This competition and consumption generate feedbacks in the complex network of biological species.

The dynamics of predator–prey systems are one of the stimulating mathematical models.[11,16–20] According to a widely accepted approach, the spatio-temporal dynamics of a predator–prey system can be expressed by the following system[21] :

$$\begin{aligned} U(X,T)_T &= \mathcal{D}U_{XX} + F(U)U - G(U)V, \\ V(X,T)_T &= \mathcal{D}V_{XX} + \kappa G(U)V - H(V)V \end{aligned} \tag{1}$$

where U and V are the densities of the prey and predator at location X and time T, respectively, $F(U)$ is the person per capita growth rate, $G(U)V$ predation, κ food utilization coefficient, $H(V)$ predator mortality per capitarate and $\mathcal{D}$ diffusion coefficient.

For different species, the functions F, G and H can be of different types. Here we focus on the Allee effect of prey dynamics is exposed,

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https://doi.org/10.1016/j.padiff.2024.100785
Received 20 May 2024; Received in revised form 11 June 2024; Accepted 28 June 2024
Available online 2 July 2024

so that the per capita growth rate is not a monotonically decreasing function of prey density, but has a local maximum. Accordingly, the standard parametrization is

$$F(U) = \alpha(U - U_0)(K - U), \tag{2}$$

where K is the prey carrying capacity and U_0 is a specific measure of the Allee effect. Here, the Allee effect is called strong when $U_0 > 0$ and weak when $-K < U_0 \leq 0$.[21] On the other hand, we assume that the per capita predator mortality is defines as

$$H(V) = M + d_0 V^n, \tag{3}$$

where M, d_0 and n are positive constants. Within the scope of the study, $n = 2$ will be considered. Finally, the predator we assume that it has a linear response to its prey, i.e. $G(U) = \eta U$. Then using (1) with (2) and (3) we get;

$$\begin{aligned} U(X,T)_T &= \mathcal{D}U_{XX} + \alpha(U - U_0)(K - U)U - \eta UV \\ V(X,T)_T &= \mathcal{D}V_{XX} + \kappa\eta UV - MV - d_0 V^3. \end{aligned} \tag{4}$$

Here if $u = \dfrac{U}{K}$, $v = \dfrac{\eta V}{\alpha K^2}$, $x = X\sqrt{\dfrac{\alpha K^2}{\mathcal{D}}}$, $t = T\alpha K^2$ dimensionless variable transformation is applied and if $\Omega = U_0 K^{-1}$, $k = \kappa\eta(\alpha K)^{-1}$, $m = M(\alpha K^2)^{-1}$ and $\delta = d_0 \alpha K^2 \eta^{-2}$ simplifications are made, for $-\infty < x < \infty$, $t > 0$; following predator–prey system is obtained.

$$\begin{aligned} u_t &= u_{xx} - \Omega u + (1+\Omega)u^2 - u^3 - uv \\ v_t &= v_{xx} + kuv - mv - \delta v^3, \end{aligned} \tag{5}$$

where Ω, k, m, δ are positive constants. The prey and predator densities at position x and time t, respectively, are represented by the functions u and v. The predator population relies entirely on the abundance of prey, given that the sole factor in the predator equation contributing to population growth is the term kuv, which signifies the increase in birth rate resulting from prey consumption. Parameters Ω and m denote the individual mortality rates of prey and predators, respectively, within the linear, small population scenario. Additionally, the term δv^3 represents a closure relationship that incorporates the influences of higher trophic levels.[11,21,22]

This study's primary goal is to investigate some of the most well-known definitions of fractional derivatives for predator–prey system (5), such as conformable,[23] M-truncated,[24] and β.[25] Since fractional derivative operators may represent non-local and non-linear behaviours, they are useful in research and engineering. These features make them useful for modelling intricate phenomena like viscoelasticity, anomalous diffusion, and fractional-order control systems. They offer a more realistic portrayal of systems displaying power-law behaviours or fractal-like features in mathematical modelling. They are also crucial to quantum mechanics, porous media flow, and diffusion processes in physics. Compared to traditional derivatives, fractional derivatives provide a more precise way of describing complicated dynamics and phenomena. In order to improve the model's ability to describe the physical direction, we use the fractional derivative operators that we previously covered to represent the equation. Our opinion is that the exact solutions we were able to arrive at for every fractional derivative operator are novel solutions that have not been found in any previous research, and that both these solutions and the comparisons we were able to make across fractional derivative operators are significant contributions. The novelty of this work consists on applying the Bäcklund transform approach and the generalized exponential function method to produce accurate analytical solutions, along with the usage of conformal, M-truncated, and Beta time fractional derivative operators for a physical interpretation. By enhancing our knowledge of predator–prey systems through the Allee effect, we think that our study which is the first use of these methodologies in this model—offers distinct outcomes from the body of previous researches.

This work consists of seven sections. We have provided some essential definitions and theorems in the second section. In the second section we also explain the general scheme of the generalized exponential rational function method (GERFM).[26,27] In the next section, we redefine (5) for the conformable, M-truncated, and Beta derivatives. In this section also, we reduced the (5) to an ordinary differential equation (ODE) by applying the transformations given separately for every operators. In the fourth section, we analyse exact solutions of time fractional predator–prey system via GERFM. Using Bäcklund transformation[28,29] we obtain different types of soliton solutions of the system (5) in the next section. Graphical representations of some of the obtained solutions, physical interpretations and comparison of derivative operators are given in section six. Finally in the "Conclusion" we summarize our results, highlighting the novelty of the solutions.

2. Mathematical methodologies and background

This section focuses on the key features and formulas of the derivative operators we intend to discuss that are well-automated. Compared to the other standard fractional derivative operators, they offer a number of significant advantages. On the other hand, we offer the GERFM's primary framework.

2.1. Conformable derivative

Refs. 30–32 contains notable studies on the conformable derivative.

Definition 2.1 (*Refs. 23, 30*). Let $\omega : [\eth_1, \eth_2] \times (0,\infty) \to \mathbb{R}$. Next, we define the conformable fractional derivative (CFD) of Ω as,

$$tD_\alpha(\omega)(x,t) = \lim_{\xi \to 0} \frac{\omega(x, t + \xi t^{1-\alpha}) - \omega(x,t)}{\xi}, \tag{6}$$

where $t > 0$ and $\alpha \in (0,1]$.

Theorem 2.1 (*Ref. 23*). *If a function $\omega : [0,\infty] \to \mathbb{R}$ is α differentiable ($\alpha \in (0,1]$) at $t_0 > 0$, then ω is continuous at t_0.*

Definition 2.2 (*Ref. 30*). Let $\omega : [\eth_1, \eth_2] \times (0,\infty) \to \mathbb{R}$, then ω's (left) CFD, beginning from $\eth_1$, is described as followed:

$$tD_\alpha^{\eth_1}(\omega)(x,t) = \lim_{\xi \to 0} \frac{\omega(x, t + \xi(t-\alpha)^{1-\alpha}) - \omega(x,t)}{\xi}, \tag{7}$$

$\alpha \in (0,1]$, for all $t > 0$.

Clearly, when $\alpha = 0$ we have $tD_\alpha^{\eth_1}(\omega)(x,t) = \omega(x,t)$.

We shall provide a few conformable derivative properties. In this case, the term α-differentiable denotes the existence of a CFD of order α for ω.

Theorem 2.2 (*Refs. 23, 33*). *Let $\alpha \in (0,1]$ and ω_1 and ω_2 are α-differentiable at $t > 0$, then*

- $D_t^\alpha(\mu_1\omega_1(t) + \mu_2\omega_2(t)) = \mu_1 D_t^\alpha(\omega_1(t)) + \mu_2 D_t^\alpha(\omega_2(t))$ *for all* $\mu_1, \mu_2 \in \mathbb{R}$,
- $D_t^\alpha(\omega_1(t)\omega_2(t)) = \omega_1(t)D_t^\alpha(\omega_2(t)) + \omega_2(t)D_t^\alpha(\omega_1(t))$,
- $D_t^\alpha\left(\dfrac{\omega_2(t)}{\omega_1(t)}\right) = \dfrac{\omega_1(t)D_t^\alpha(\omega_2(t)) - \omega_2(t)D_t^\alpha(\omega_1(t))}{\omega_1(t)^2}$,
- $D_t^\alpha(t^\gamma) = \gamma t^{\gamma-\alpha}$ *for all* $\gamma \in \mathbb{R}$,
- $D_t^\alpha(k\omega_1(t)) = kD_t^\alpha(\omega_1(t))$, *where k is a constant;*
- $D_t^\alpha(\omega_1(t)) = t^{1-\alpha}\omega_1'(t)$.

2.2. M-truncated derivative

Definition 2.3. Mittag-Leffler function[33,34] with one parameter given as,

$$\mathbb{E}_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + 1)}. \tag{8}$$

the abbreviated form of it says

$$j\mathbb{E}_\beta(z) = \sum_{k=0}^{j} \frac{z^k}{\Gamma(\beta k + 1)}, \tag{9}$$

where $\beta > 0$ and $z \in \mathbb{C}$.

Definition 2.4 (*Refs. 24, 33, 34*). Let $\omega : [0,\infty) \to \mathbb{R}$. Following that, the derivative of ω that is α-th order M−truncated is defined as

$$jD_M^{\alpha,\beta}(\omega(t)) = \lim_{\xi\to 0} \frac{\omega(t + j\mathbb{E}_\beta(\xi t^{-\alpha})) - \omega(t)}{\xi}, \tag{10}$$

where $t > 0$ and $j\mathbb{E}_\beta(.), \beta > 0$ is the truncated Mittag-Leffler function.

Theorem 2.3 (*Ref. 33*). *Let $\alpha \in (0,1]$, $\beta > 0$ and ω_1 and ω_2 are α-differentiable at $t > 0$, then*

- $jD_M^{\alpha,\beta}(\mu_1\omega_1(t) + \mu_2\omega_2(t)) = \mu_1 jD_M^{\alpha,\beta}(\omega_1(t)) + \mu_2 jD_M^{\alpha,\beta}(\omega_2(t))$ *for all* $\mu_1, \mu_2 \in \mathbb{R}$,
- $jD_M^{\alpha,\beta}(\omega_1(t)\omega_2(t)) = \omega_1(t) jD_M^{\alpha,\beta}(\omega_2(t)) + \omega_2(t) jD_M^{\alpha,\beta}(\omega_1(t))$,
- $jD_M^{\alpha,\beta}\left(\frac{\omega_2(t)}{\omega_1(t)}\right) = \frac{\omega_1(t) jD_M^{\alpha,\beta}(\omega_2(t)) - \omega_2(t) jD_M^{\alpha,\beta}(\omega_1(t))}{\omega_1(t)^2}$,
- $jD_M^{\alpha,\beta}(t^\gamma) = \gamma t^{\gamma-\alpha}$ *for all* $\gamma \in \mathbb{R}$,
- $jD_M^{\alpha,\beta}(k\omega_1(t)) = k jD_M^{\alpha,\beta}(\omega_1(t))$, *where k is a constant;*
- $jD_M^{\alpha,\beta}(\omega_1(t)) = \frac{t^{1-\alpha}}{\Gamma(\beta+1)}\omega_1'(t)$.

2.3. Beta derivative

Definition 2.5 According to Refs. 25, 33, a beta derivative has been defined by the following definition:

$${}_0^A D_x^\alpha(L(X)) = \lim_{\xi\to 0} \frac{L\left(x + \frac{1}{\Gamma(\alpha)}\right) - L(x)}{\xi}. \tag{11}$$

Theorem 2.4 (*Refs. 25, 33*). *Let $\alpha \in (0,1]$, and ω_1 and ω_2 are α-differentiable at $t > 0$, then*

- ${}_0^A D_x^\alpha(\mu_1\omega_1(x) + \mu_2\omega_2(x)) = \mu_1 {}_0^A D_x^\alpha(\omega_1(x)) + \mu_2 {}_0^A D_x^\alpha(\omega_2(x))$ *for all* $\mu_1, \mu_2 \in \mathbb{R}$,
- ${}_0^A D_x^\alpha(\omega_1(x)\omega_2(x)) = \omega_1(x) {}_0^A D_x^\alpha(\omega_2(x)) + \omega_2(x) {}_0^A D_x^\alpha(\omega_1(x))$,
- ${}_0^A D_x^\alpha\left(\frac{\omega_2(x)}{\omega_1(x)}\right) = \frac{\omega_1(x) {}_0^A D_x^\alpha(\omega_2(x)) - \omega_2(x) {}_0^A D_x^\alpha(\omega_1(x))}{\omega_1(x)^2}$,
- ${}_0^A D_x^\alpha(\Phi) = 0$, *$\Phi$ is constant,*
- ${}_0^A D_x^\alpha(\omega_1(x)) = \left(x + \frac{1}{\Gamma(\alpha)}\right)^{1-\alpha}\omega_1'(x)$.

2.4. The generalized exponential rational function method (GERFM)

The unknown coefficients are generated by the algorithm's stages, and the exact solution is seen as a finite series expansion in this analytical solution finding architecture. In this method, unlike other exact solution methods, no auxiliary equations are used. The basic stages of this approach are shown.[26,27] Let us examine the one that follows PDE in this manner:

$$\kappa(\tau, \tau_x, \tau_t, \tau_{xx}, \tau_{tt}, \ldots) = 0. \tag{12}$$

In (12), using the travelling wave variable $\xi = K(x - ct)$, (c is wave speed) and transformation $\tau(x,t) = \tau(\xi)$ converts Eq. (12) into the following ordinary differential equation (ODE)

$$S(\tau, \tau', \tau', \tau'', \tau'', \ldots) = 0. \tag{13}$$

Step 1:
We consider the following finite series expansion to be the solution to Eq. (13):

$$\tau = A_0 + \sum_{i=1}^{N} A_i \Xi^i(\xi) + \sum_{i=1}^{N} B_i \Xi^{-i}(\xi), \tag{14}$$

where

$$\Xi(\xi) = \frac{a_1 e^{b_1\xi} + a_2 e^{b_2\xi}}{a_3 e^{b_3\xi} + a_4 e^{b_4\xi}}, \tag{15}$$

and a_i, b_i $(i = 1,2,3,4)$, A_0, A_i, B_i are constants to be determined, and also N is evaluated with balancing principle.

Step 2:
After substituting (15) for in Eq. (13), collecting $e^{b_i}, i = 1,2,3,4$ generates a system of algebraic equations. The coefficients of $e^{b_i}, i = 1,2,3,4$ equate to zero and the constants a_i, b_i $(i = 1,2,3,4)$, A_0, A_i, B_i are then determined by solving the system.

3. Governing equation

In this part, we give the fractional predator–prey system (Eq. (5)) in relation to several definitions of derivatives. **I**:The conformable derivative with respect to t allows us to express Eq. (5) in the following manner.

$$\begin{aligned} D_t^\alpha u &= u_{xx} - \Omega u + (1+\Omega)u^2 - u^3 - uv \\ D_t^\alpha v &= v_{xx} + kuv - mv - \delta v^3, \end{aligned} \tag{16}$$

where $0 < \alpha \le 1$.

II: Using the M-truncated with respect to t, we may write the Eq. (5) as follows, where $\alpha \in (0,1]$,

$$\begin{aligned} jD_{M,t}^{\alpha,\beta^\alpha} u &= u_{xx} - \Omega u + (1+\Omega)u^2 - u^3 - uv \\ jD_{M,t}^{\alpha,\beta} v &= v_{xx} + kuv - mv - \delta v^3, \end{aligned} \tag{17}$$

where $0 < \alpha \le 1$.

III: In Beta fractional derivative with respect to t, we can give the Eq.(5) as follows:

$$\begin{aligned} {}_0^A D_t^\alpha u &= u_{xx} - \Omega u + (1+\Omega)u^2 - u^3 - uv \\ {}_0^A D_t^\alpha v &= v_{xx} + kuv - mv - \delta v^3. \end{aligned} \tag{18}$$

3.1. Transformations and reduced equation

To solve Eqs. (16), (17), and (18) we will take use of the wave transforms shown below.

$$\begin{aligned} u(x,t) &= H(\xi) \\ v(x,t) &= G(\xi) \end{aligned} \tag{19}$$

According to the each derivative definitions that were given in the above, ξ is given below in several kinds.

- According to CFD,we have
$$\xi = x - \left(\frac{c}{\alpha}t^\alpha\right), \tag{20}$$
- By means of M-truncated derivative,we take
$$\xi = x - \frac{\Gamma(\beta+1)}{\alpha}(ct^\alpha), \tag{21}$$
- For Beta derivative, we use
$$\xi = x - \frac{c}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha. \tag{22}$$

The transformations indicated above help reduce the Eqs. (16)–(18) to an ODE. To do that we apply[11] the subsequent relationships between the parameters of the equation:

$$m = \Omega \text{ and } k + \frac{1}{\sqrt{\delta}} = 1 + \Omega,$$

then using (19)–(22) we obtain,

$$\begin{aligned} &H'' + cH' - \Omega H + \left(k + \frac{1}{\sqrt{\delta}}\right)H^2 - H^3 - HG = 0 \\ &G'' + cG' + kHG - mG - \delta G^3 = 0. \end{aligned} \tag{23}$$

Assuming, $G(\xi) = \frac{H(\xi)}{\sqrt{\delta}}$ we reach to reduced ODE,

$$H'' + cH' + kH^2 - mH - H^3 = 0. \tag{24}$$

.

4. Analysis of solutions via GERFM

This section aims to provide several travelling wave solutions to the predator–prey system (16)–(18) by applying the GERFM approach. As a result of balancing the highest order derivative and the highest degree nonlinear term in (24), we conclude that the order of the finite series expansion in Eq. (14) is 1.

Case 1: In this case we consider $a_i = [-2, -3, 1, 1]$ and $b_i = [1, 0, 1, 0]$. Then Eq. (15) turns into

$$\Xi(\xi) = \frac{-2e^{\xi} - 3}{e^{\xi} + 1}. \tag{25}$$

Substituting Eq. (14) with Eq. (25) into (24) for determine unknown constants, and then follow the steps of GERFM, we yield following solution sets:

Set 1:

$$\left\{c = k\sqrt{2} - 3, m = k\sqrt{2} - 2, A_0 = 3\sqrt{2}, A_1 = \sqrt{2}, B_1 = 0\right\}$$

Considering these values, along with (14) and (25), gives

$$H(\xi) = 3\sqrt{2} + \frac{\sqrt{2}\,\left(-2e^{\xi} - 3\right)}{e^{\xi} + 1}, \tag{26}$$

and so we get solution of (16) for CFD using (20) as,

$$\begin{aligned} u_1(x,t) &= \frac{\sqrt{2}\,e^{\frac{-\sqrt{2}\,t^{\alpha}k + x\alpha + 3t^{\alpha}}{\alpha}}}{e^{\frac{-\sqrt{2}\,t^{\alpha}k + x\alpha + 3t^{\alpha}}{\alpha}} + 1} \\ v_1(x,t) &= \frac{\sqrt{2}\,e^{\frac{-\sqrt{2}\,t^{\alpha}k + x\alpha + 3t^{\alpha}}{\alpha}}}{\left(e^{\frac{-\sqrt{2}\,t^{\alpha}k + x\alpha + 3t^{\alpha}}{\alpha}} + 1\right)\sqrt{\frac{1}{k\sqrt{2} - 1 - k}}}. \end{aligned} \tag{27}$$

Similarly, we expressed the solutions in beta and M-truncated fractional derivatives and presented them in the appendix.

Set 2:

$$\left\{c = A_0\sqrt{2} - 5, m = A_0^2 - 5A_0\sqrt{2} + 12, k = 2A_0 - 5\sqrt{2}, A_1 = 0, B_1 = 6\sqrt{2}\right\}$$

which yields,

$$H(\xi) = A_0 + \frac{6\sqrt{2}\,(e^{\epsilon} + 1)}{-2e^{\epsilon} - 3}, \tag{28}$$

and so,

$$\begin{aligned} u_2(x,t) &= \frac{\left(2A_0 - 6\sqrt{2}\right)e^{\frac{-t^{\alpha}\sqrt{2}\,A_0 + x\alpha + 5t^{\alpha}}{\alpha}} + 3A_0 - 6\sqrt{2}}{2e^{\frac{-t^{\alpha}\sqrt{2}\,A_0 + x\alpha + 5t^{\alpha}}{\alpha}} + 3} \\ v_2(x,t) &= \frac{2\left(\left(A_0 - 3\sqrt{2}\right)e^{\frac{-t^{\alpha}\sqrt{2}\,A_0 + x\alpha + 5t^{\alpha}}{\alpha}} + \frac{3A_0}{2} - 3\sqrt{2}\right)}{\sqrt{\frac{1}{(-5A_0 + 5)\sqrt{2} + A_0^2 - 2A_0 + 13}}\left(2e^{\frac{-t^{\alpha}\sqrt{2}\,A_0 + x\alpha + 5t^{\alpha}}{\alpha}} + 3\right)}. \end{aligned} \tag{29}$$

Case 2: If one considers $a_i = [1-i, -1-i, -1, 1]$ and $b_i = [i, -i, i, -i]$, then Eq. (15) will convert into

$$\Xi(\xi) = \frac{\cos(\xi) - \sin(\xi)}{\sin(\xi)}. \tag{30}$$

Set 1:

$$\left\{k = 2\left(\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right)^3 + 2\sqrt{2} + 2i\sqrt{2}, m = 4\left(\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right)^2 - 4, c = -2, A_0 = \sqrt{2} + i\sqrt{2}, A_1 = 0, B_1 = 2\left(\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right)^3 + \sqrt{2} + i\sqrt{2}\right\}$$

Keeping these values in mind, along with Eqs. (14) and (30), yields the following result:

$$H(\xi) = \frac{\sqrt{2}\,\left(2i\cos(\epsilon)^2 + 2\sin(\epsilon)\cos(\epsilon) + 1 - i\right)}{2\cos(\epsilon)^2 - 1}. \tag{31}$$

Thus, we attain a travelling wave solution for the conformable system (16) as,

$$\begin{aligned} u_3(x,t) &= \left(1 + i - \frac{2\sin\left(\frac{x\alpha + 2t^{\alpha}}{\alpha}\right)}{-\cos\left(\frac{x\alpha + 2t^{\alpha}}{\alpha}\right) + \sin\left(\frac{x\alpha + 2t^{\alpha}}{\alpha}\right)}\right)\sqrt{2} \\ v_3(x,t) &= \frac{\left((1+i)\cos\left(\frac{x\alpha + 2t^{\alpha}}{\alpha}\right) + (1-i)\sin\left(\frac{x\alpha + 2t^{\alpha}}{\alpha}\right)\right)\sqrt{2}}{\sqrt{-\frac{1}{3 - 4i + (1+3i)\sqrt{2}}}\left(\cos\left(\frac{x\alpha + 2t^{\alpha}}{\alpha}\right) - \sin\left(\frac{x\alpha + 2t^{\alpha}}{\alpha}\right)\right)}. \end{aligned} \tag{32}$$

Set 2:

$$\left\{c = -2, k = 2\sqrt{2}, m = 4, A_0 = 0, A_1 = 0, B_1 = -2\sqrt{2}\right\}$$

In light of these values, along with Eqs. (14) and (30), yields the following result:

$$H(\xi) = -\frac{2\sqrt{2}\,\sin(\xi)}{\cos(\xi) - \sin(\xi)}. \tag{33}$$

Thus, we acquire the following exact solution for the conformable system (16) as

$$\begin{aligned} u_4(x,t) &= \frac{2\sqrt{2}\,\sin\left(\frac{x\alpha + 2t^{\alpha}}{\alpha}\right)}{-\cos\left(\frac{x\alpha + 2t^{\alpha}}{\alpha}\right) + \sin\left(\frac{x\alpha + 2t^{\alpha}}{\alpha}\right)} \\ v_4(x,t) &= \frac{2\sqrt{2}\,\sin\left(\frac{x\alpha + 2t^{\alpha}}{\alpha}\right)\sqrt{5 - 2\sqrt{2}}}{-\cos\left(\frac{x\alpha + 2t^{\alpha}}{\alpha}\right) + \sin\left(\frac{x\alpha + 2t^{\alpha}}{\alpha}\right)}. \end{aligned} \tag{34}$$

Case 3: In this case we consider $a_i = [-2 - i, 2 - i, -1, 1]$ and $b_i = [i, -i, i, -i]$, then Eq. (15) turns into

$$\Xi(\xi) = \frac{cos(\xi) + 2sin(\xi)}{sin(\xi)}. \tag{35}$$

Substituting Eq. (14) with Eq. (35) and then follow the steps above to obtain following solution sets:

Set 1:

$$\left\{c = 7, k = 7\sqrt{2}, m = 53/2, A_0 = \frac{3\sqrt{2}}{2}, A_1 = 0, B_1 = 5\sqrt{2}\right\}$$

Considering these values, along with (14) and (35), gives

$$H(\xi) = \frac{\sqrt{2}\,(3\cos(\epsilon) + 16\sin(\epsilon))}{2\cos(\epsilon) + 4\sin(\epsilon)}, \tag{36}$$

and so we get solution of (16) for CFD using (20) as,

$$\begin{aligned} u_5(x,t) &= \frac{\sqrt{2}\,\left(3\cos\left(\frac{-x\alpha + 7t^{\alpha}}{\alpha}\right) - 16\sin\left(\frac{-x\alpha + 7t^{\alpha}}{\alpha}\right)\right)}{2\cos\left(\frac{-x\alpha + 7t^{\alpha}}{\alpha}\right) - 4\sin\left(\frac{-x\alpha + 7t^{\alpha}}{\alpha}\right)} \\ v_5(x,t) &= \frac{\sqrt{2}\,\left(3\cos\left(\frac{-x\alpha + 7t^{\alpha}}{\alpha}\right) - 16\sin\left(\frac{-x\alpha + 7t^{\alpha}}{\alpha}\right)\right)\sqrt{110 - 28\sqrt{2}}}{4\cos\left(\frac{-x\alpha + 7t^{\alpha}}{\alpha}\right) - 8\sin\left(\frac{-x\alpha + 7t^{\alpha}}{\alpha}\right)}. \end{aligned} \tag{37}$$

Set 2:

$$\left\{c = 7, k = \frac{13\sqrt{2}}{4} + \frac{13i\sqrt{2}}{8} + \frac{\left(2\sqrt{2} + i\sqrt{2}\right)^3}{16}, m = \frac{7\left(2\sqrt{2} + i\sqrt{2}\right)^2}{4} - \frac{29}{2}, A_0 = 2\sqrt{2} + i\sqrt{2}, A_1 = 0, B_1 = \frac{\left(2\sqrt{2} + i\sqrt{2}\right)^3}{8} - \frac{11\sqrt{2}}{2} - \frac{11i\sqrt{2}}{4}\right\}$$

Keeping these values in mind, along with Eqs. (14) and (35), yields the following result:

$$H(\xi) = \frac{\sqrt{2}\,\left(5i\cos(\xi)^2 - 5\sin(\xi)\cos(\xi) + 2 - 4i\right)}{5\cos(\xi)^2 - 4}. \tag{38}$$

Thus, we derive an exact solution for the conformable system (16) as

$$\begin{aligned} u_6(x,t) &= \left(2+\mathrm{i}-\frac{5\sin\left(\frac{-x\alpha+7t^{\alpha}}{\alpha}\right)}{-\cos\left(\frac{-x\alpha+7t^{\alpha}}{\alpha}\right)+2\sin\left(\frac{-x\alpha+7t^{\alpha}}{\alpha}\right)}\right)\sqrt{2} \\ v_6(x,t) &= \frac{(2+\mathrm{i})\cos\left(\frac{-x\alpha+7t^{\alpha}}{\alpha}\right)+(1-2\,\mathrm{i})\sin\left(\frac{-x\alpha+7t^{\alpha}}{\alpha}\right)}{\sqrt{-\frac{1}{6-28\,\mathrm{i}+(7+6\,\mathrm{i})\sqrt{2}}}\left(\cos\left(\frac{-x\alpha+7t^{\alpha}}{\alpha}\right)-2\sin\left(\frac{-x\alpha+7t^{\alpha}}{\alpha}\right)\right)}. \end{aligned} \tag{39}$$

Case 4: In this case we consider $a_i=[1,-3,-1,1]$ and $b_i=[1,-1,1,-1]$, then Eq. (15) becomes

$$\Xi(\xi)=\frac{cosh(\xi)-2sinh(\xi)}{sinh(\xi)}. \tag{40}$$

Substituting Eq. (14) with Eq. (40) and then follow the steps above to obtain following solution sets.

Set 1:

$$\left\{c=-9,k=9\sqrt{2},m=77/2,A_0=\frac{5\sqrt{2}}{2},A_1=0,B_1=3\sqrt{2}\right\}$$

Considering these values, along with (14) and (40), gives

$$H(\xi)=\frac{\sqrt{2}\,(5\cosh(\epsilon)-16\sinh(\epsilon))}{2\cosh(\epsilon)-4\sinh(\epsilon)}, \tag{41}$$

and so we get solution of (16) for CFD using (20) as,

$$\begin{aligned} u_7(x,t) &= \frac{5\sqrt{2}}{2}-\frac{3\sqrt{2}\sinh\left(x+\frac{9t^{\alpha}}{\alpha}\right)}{\cosh\left(x+\frac{9t^{\alpha}}{\alpha}\right)-2\sinh\left(x+\frac{9t^{\alpha}}{\alpha}\right)} \\ v_7(x,t) &= \left(\frac{5\sqrt{2}}{2}-\frac{3\sqrt{2}\sinh\left(x+\frac{9t^{\alpha}}{\alpha}\right)}{\cosh\left(x+\frac{9t^{\alpha}}{\alpha}\right)-2\sinh\left(x+\frac{9t^{\alpha}}{\alpha}\right)}\right)\sqrt{\frac{79}{2}-9\sqrt{2}}. \end{aligned} \tag{42}$$

Set 2:

$$\left\{c=-9,k=\frac{15\sqrt{2}}{2},m=22,A_0=3\sqrt{2},A_1=0,B_1=3\sqrt{2}\right\}$$

Considering these values, along with Eq. (14) and (40), yields the following result:

$$H(\xi)=3\sqrt{2}+\frac{3\sqrt{2}\sinh(\epsilon)}{\cosh(\epsilon)-2\sinh(\epsilon)}. \tag{43}$$

Thus, we secure a wave solution for the conformable system (16) as

$$\begin{aligned} u_8(x,t) &= 3\sqrt{2}+\frac{3\sqrt{2}\sinh\left(x+\frac{9t^{\alpha}}{\alpha}\right)}{\cosh\left(x+\frac{9t^{\alpha}}{\alpha}\right)-2\sinh\left(x+\frac{9t^{\alpha}}{\alpha}\right)} \\ v_8(x,t) &= \left(3\sqrt{2}+\frac{3\sqrt{2}\sinh\left(x+\frac{9t^{\alpha}}{\alpha}\right)}{\cosh\left(x+\frac{9t^{\alpha}}{\alpha}\right)-2\sinh\left(x+\frac{9t^{\alpha}}{\alpha}\right)}\right)\sqrt{23-\frac{15\sqrt{2}}{2}}. \end{aligned} \tag{44}$$

Case 5: In this case we consider $a_i=[0,-1,1,1]$ and $b_i=[0,0,1,0]$, then Eq. (15) turn into

$$\Xi(\xi)=\frac{-1}{e^{-\xi}+1}. \tag{45}$$

Substituting Eq. (14) with (45) and then follow the steps above to obtain following solution sets.

Set 1:

$$\left\{c=k\sqrt{2}-3,m=k\sqrt{2}-2,A_0=\sqrt{2},A_1=\sqrt{2},B_1=0\right\}$$

Keeping these values in mind, along with (14) and (45), gives

$$H(\xi)=\frac{\sqrt{2}\,\mathrm{e}^{\xi}}{\mathrm{e}^{\xi}+1} \tag{46}$$

and so we get solution of (16) for CFD using (20) as,

$$\begin{aligned} u_9(x,t) &= \frac{\sqrt{2}\,\mathrm{e}^{\frac{-t^{\alpha}\sqrt{2}k+x\alpha+3t^{\alpha}}{\alpha}}}{\mathrm{e}^{\frac{-t^{\alpha}\sqrt{2}k+x\alpha+3t^{\alpha}}{\alpha}}+1} \\ v_9(x,t) &= \frac{\sqrt{2}\,\mathrm{e}^{\frac{-t^{\alpha}\sqrt{2}k+x\alpha+3t^{\alpha}}{\alpha}}}{\left(\mathrm{e}^{\frac{-t^{\alpha}\sqrt{2}k+x\alpha+3t^{\alpha}}{\alpha}}+1\right)\sqrt{\frac{1}{k\sqrt{2}-1-k}}}. \end{aligned} \tag{47}$$

Set 2:

$$\left\{c=A_0\sqrt{2}-3,k=2A_0-3\sqrt{2},m=A_0^2-3A_0\sqrt{2}+4,A_1=0,B_1=2\sqrt{2}\right\}$$

Keeping these values in mind, along with Eqs. (14) and (45), yields the following result:

$$H(\xi)=\frac{\left(-2\mathrm{e}^{\xi}-2\right)\sqrt{2}+A_0\left(\mathrm{e}^{\xi}+2\right)}{\mathrm{e}^{\xi}+2}. \tag{48}$$

Thus, we will determine a wave solution for the conformable system (16) as

$$\begin{aligned} u_{10}(x,t) &= \frac{\left(A_0-2\sqrt{2}\right)\mathrm{e}^{\frac{-t^{\alpha}\sqrt{2}A_0+x\alpha+3t^{\alpha}}{\alpha}}+2A_0-2\sqrt{2}}{\mathrm{e}^{\frac{-t^{\alpha}\sqrt{2}A_0+x\alpha+3t^{\alpha}}{\alpha}}+2} \\ v_{10}(x,t) &= \frac{\left(A_0-2\sqrt{2}\right)\mathrm{e}^{\frac{-t^{\alpha}\sqrt{2}A_0+x\alpha+3t^{\alpha}}{\alpha}}+2A_0-2\sqrt{2}}{\sqrt{\frac{1}{(-3A_0+3)\sqrt{2}+A_0^2-2A_0+5}}\left(\mathrm{e}^{\frac{-t^{\alpha}\sqrt{2}A_0+x\alpha+3t^{\alpha}}{\alpha}}+2\right)}. \end{aligned} \tag{49}$$

Case 6: In this case we consider $a_i=[-12,-2,1,1]$ and $b_i=[1,0,1,0]$, then Eq. (15) turn into

$$\Xi(\xi)=\frac{-e^{\xi}-2}{e^{\xi}+1}. \tag{50}$$

Substituting Eq. (14) with (50) and then follow the steps above to obtain following solution sets.

Set 1:

$$\left\{c=k\sqrt{2}-3,m=k\sqrt{2}-2,A_0=2\sqrt{2},A_1=\sqrt{2},B_1=0\right\}$$

Keeping these values in mind, along with (14) and (50), gives

$$H(\xi)=2\sqrt{2}+\frac{\sqrt{2}\left(-\mathrm{e}^{\xi}-2\right)}{\mathrm{e}^{\xi}+1}, \tag{51}$$

and so we get solution of (5) for CFD using (20) as,

$$\begin{aligned} u_{11}(x,t) &= 2\sqrt{2}+\frac{\sqrt{2}\left(-\mathrm{e}^{x-\frac{\left(\sqrt{2}k-3\right)t^{\alpha}}{\alpha}}-2\right)}{\mathrm{e}^{x-\frac{\left(\sqrt{2}k-3\right)t^{\alpha}}{\alpha}}+1} \\ v_{11}(x,t) &= \frac{\sqrt{2}\,\mathrm{e}^{\frac{-\sqrt{2}t^{\alpha}k+x\alpha+3t^{\alpha}}{\alpha}}}{\left(\mathrm{e}^{\frac{-\sqrt{2}t^{\alpha}k+x\alpha+3t^{\alpha}}{\alpha}}+1\right)\sqrt{\frac{1}{\sqrt{2}k-1-k}}}. \end{aligned} \tag{52}$$

Set 2:

$$\left\{c=A_0\sqrt{2}-3,k=2A_0-3\sqrt{2},m=A_0^2-3A_0\sqrt{2}+4,A_1=0,B_1=2\sqrt{2}\right\}$$

which yields,

$$H(\xi)=\frac{\left(-2\,\mathrm{e}^{\epsilon}-2\right)\sqrt{2}+A_0\left(\mathrm{e}^{\xi}+2\right)}{\mathrm{e}^{\xi}+2}, \tag{53}$$

and so,

$$u_{12}(x,t)=\frac{\left(A_0-2\sqrt{2}\right)e^{\frac{-t^\alpha\sqrt{2}A_0+x\alpha+3t^\alpha}{\alpha}}+2A_0-2\sqrt{2}}{e^{\frac{-t^\alpha\sqrt{2}A_0+x\alpha+3t^\alpha}{\alpha}}+2}$$

$$v_{12}(x,t)=\frac{\left(A_0-2\sqrt{2}\right)e^{\frac{-t^\alpha\sqrt{2}A_0+x\alpha+3t^\alpha}{\alpha}}+2A_0-2\sqrt{2}}{\sqrt{\frac{1}{(-3A_0+3)\sqrt{2}+A_0^2-2A_0+5}}\left(e^{\frac{-t^\alpha\sqrt{2}A_0+x\alpha+3t^\alpha}{\alpha}}+2\right)}. \tag{54}$$

Case 7: In this case we consider $a_i=[i,-i,1,1]$ and $b_i=[i,-i,i,-i]$, then Eq. (15) converts into

$$\Xi(\xi)=-\frac{\sin(\xi)}{\cos(\xi)}. \tag{55}$$

Substituting Eq. (14) with (55) into Eq. (24) and then follow the steps above to obtain following solution sets.

Set 1:

$$\left\{c=\frac{i}{2}(m+4), k=-\frac{i}{4}\sqrt{2}(m-8), A_0=i\sqrt{2}, A_1=0, B_1=\sqrt{(2)}\right\}$$

Keeping these values in mind, along with (14) and (55), gives

$$H(\xi)=\sqrt{2}\,(i-\cot(\epsilon)) \tag{56}$$

and so we get solution of (16) for CFD using (20) as,

$$u_{13}(x,t)=\sqrt{2}\left(i+\cot\left(\frac{i(m+4)t^\alpha-2x\alpha}{2\alpha}\right)\right)$$

$$v_{13}(x,t)=\frac{\sqrt{2}\left(i+\cot\left(\frac{i(m+4)t^\alpha-2x\alpha}{2\alpha}\right)\right)}{2\sqrt{\frac{1}{4+i\sqrt{2}(m-8)+4m}}} \tag{57}$$

Set 2:

$$\left\{c=\sqrt{2}\sqrt{m-2}, k=2\sqrt{m-2}, A_0=\sqrt{m-2}, A_1=0, B_1=\sqrt{2}\right\}$$

which yields,

$$H(\xi)=\sqrt{m-2}-\sqrt{2}\cot(\epsilon), \tag{58}$$

and so,

$$u_{14}(x,t)=\sqrt{m-2}+\sqrt{2}\cot\left(\frac{w\,t^\alpha-x\alpha}{\alpha}\right)$$

$$v_{14}(x,t)=\frac{\sqrt{m-2}+\sqrt{2}\cot\left(\frac{w\,t^\alpha-x\alpha}{\alpha}\right)}{\sqrt{\frac{1}{m-2\sqrt{m-2}+1}}}. \tag{59}$$

Set 3:

$$\left\{c=-\frac{i}{4}(m+16), k=-\frac{i}{8}\sqrt{2}(m-32), A_0=2i\sqrt{2}, A_1=\sqrt{2}, B_1=-\sqrt{(2)}\right\}$$

which yields,

$$H(\xi)=\sqrt{2}\,(2i+2\cot(\epsilon)-\sec(\epsilon)\csc(\epsilon)) \tag{60}$$

and hence,

$$u_{15}(x,t)=\sqrt{2}\left(2i+\cot\left(\frac{i(m+16)t^\alpha+4x\alpha}{4\alpha}\right)-\tan\left(\frac{i(m+16)t^\alpha+4x\alpha}{4\alpha}\right)\right)$$

$$v_{15}(x,t)=\frac{2i+\cot\left(\frac{i(m+16)t^\alpha+4x\alpha}{4\alpha}\right)-\tan\left(\frac{i(m+16)t^\alpha+4x\alpha}{4\alpha}\right)}{2\sqrt{\frac{1}{8+i\sqrt{2}(m-32)+8m}}}. \tag{61}$$

Set 4:

$$\left\{c=\sqrt{2}\sqrt{m-8}, k=2\sqrt{m-8}, A_0=\sqrt{m-8}, A_1=\sqrt{2}, B_1=-\sqrt{(2)}\right\}$$

which yields,

$$H(\xi)=\sqrt{m-8}+(\cot(\xi)-\tan(\xi))\sqrt{2}, \tag{62}$$

and we get,

$$u_{16}(x,t)=\sqrt{m-8}-\sqrt{2}\tan\left(\frac{\sqrt{2}\sqrt{m-8}\,t^\alpha+x\alpha}{\alpha}\right)+\sqrt{2}\cot\left(\frac{\sqrt{2}\sqrt{m-8}\,t^\alpha+x\alpha}{\alpha}\right)$$

$$v_{16}(x,t)=\frac{\sqrt{m-8}-\sqrt{2}\left(\tan\left(\frac{\sqrt{2}\sqrt{m-8}\,t^\alpha+x\alpha}{\alpha}\right)-\cot\left(\frac{\sqrt{2}\sqrt{m-8}\,t^\alpha+x\alpha}{\alpha}\right)\right)}{\sqrt{\frac{1}{m-2\sqrt{m-8}+1}}}. \tag{63}$$

Case 8: In this case we take $a_i=[1,1,-1,1]$ and $b_i=[1,-1,1,-1]$, then Eq. (15) gives

$$\Xi(\xi)=-\frac{\cosh(\xi)}{\sinh(\xi)}. \tag{64}$$

Substituting Eq. (14) with (64) and then follow the steps above to obtain following solution sets.

Set 1:

$$\left\{c=-\frac{m}{2}+2, k=\frac{\sqrt{2}(m+8)}{4}, A_0=\sqrt{2}, A_1=0, B_1=\sqrt{(2)}\right\}$$

Considering these values, along with (14) and (64), gives

$$H(\xi)=-\sqrt{2}\,(-1+\tanh(\epsilon)), \tag{65}$$

and so we get solution of (16) using (20) as,

$$u_{17}(x,t)=-\sqrt{2}\left(\tanh\left(\frac{(m-4)t^\alpha+2x\alpha}{2\alpha}\right)-1\right)$$

$$v_{17}(x,t)=-\frac{\sqrt{2}\left(\tanh\left(\frac{(m-4)t^\alpha+2x\alpha}{2\alpha}\right)-1\right)}{2\sqrt{-\frac{1}{\sqrt{2}(m+8)-4m-4}}}. \tag{66}$$

Set 2:

$$\left\{c=\sqrt{2}\sqrt{m+2}, k=2\sqrt{m+2}, A_0=\sqrt{m+2}, A_1=\sqrt{2}, B_1=0\right\}$$

which yields,

$$H(\xi)=-\sqrt{2}\coth(\xi)+\sqrt{m+2}, \tag{67}$$

and so,

$$u_{18}(x,t)=\sqrt{2}\coth\left(\frac{\sqrt{2}\sqrt{m+2}\,t^\alpha-x\alpha}{\alpha}\right)+\sqrt{m+2}$$

$$v_{18}(x,t)=-\frac{\sqrt{2}\left(\tanh\left(\frac{(m-4)t^\alpha+2x\alpha}{2\alpha}\right)-1\right)}{2\sqrt{-\frac{1}{\sqrt{2}(m+8)-4m-4}}}. \tag{68}$$

Set 3:

$$\left\{c=-\frac{m}{4}+4, k=\frac{\sqrt{2}(m+32)}{8}, A_0=2\sqrt{2}, A_1=\sqrt{2}, B_1=\sqrt{(2)}\right\}$$

which yields,

$$H(\xi)=-2\sqrt{2}\left(-\frac{\operatorname{sech}(\xi)\operatorname{csch}(\xi)}{2}+\coth(\xi)-1\right), \tag{69}$$

and so,

$$\begin{aligned} u_{19}(x,t) &= -\sqrt{2}\left(\coth\left(\frac{4x\alpha+(m-16)\,t^{\alpha}}{4\alpha}\right) + \tanh\left(\frac{4x\alpha+(m-16)\,t^{\alpha}}{4\alpha}\right) - 2\right) \\ v_{19}(x,t) &= -\frac{\coth\left(\frac{4x\alpha+(m-16)t^{\alpha}}{4\alpha}\right) + \tanh\left(\frac{4x\alpha+(m-16)t^{\alpha}}{4\alpha}\right) - 2}{2\sqrt{-\frac{1}{\sqrt{2}\,(m+32)-8m-8}}}. \end{aligned} \tag{70}$$

Set 4:

$$\left\{c = \sqrt{2}\,\sqrt{m+8}, k = 2\sqrt{m+8}, A_0 = \sqrt{m+8}, A_1 = \sqrt{2}, B_1 = \sqrt{(2)}\right\}$$

which yields,

$$H(\xi) = \sqrt{m+8} + (-\coth(\epsilon) - \tanh(\epsilon))\sqrt{2}, \tag{71}$$

and hence we obtain,

$$\begin{aligned} u_{20}(x,t) &= \sqrt{m+8} + \sqrt{2}\,\coth\left(\frac{\sqrt{2}\,\sqrt{m+8}\,t^{\alpha} - x\alpha}{\alpha}\right) + \sqrt{2}\,\tanh\left(\frac{\sqrt{2}\,\sqrt{m+8}\,t^{\alpha} - x\alpha}{\alpha}\right) \\ v_{20}(x,t) &= \frac{\sqrt{m+8} + \left(\coth\left(\frac{\sqrt{2}\,\sqrt{m+8}\,t^{\alpha}-x\alpha}{\alpha}\right) + \tanh\left(\frac{\sqrt{2}\,\sqrt{m+8}\,t^{\alpha}-x\alpha}{\alpha}\right)\right)\sqrt{2}}{\sqrt{\frac{1}{m-2\sqrt{m+8}+1}}}. \end{aligned} \tag{72}$$

5. Bäcklund transformation and the soliton solutions

In this section we will analys soliton solutions of (5) with help of Bäcklund transformation.[28,29] It should be noted that there must be balancing in the model for the method to be applied. Here, we will suppose that, takes the following form in the view of the Bäcklund transformation[28,29]:

$$H = A_1 \frac{\partial^{\epsilon}\phi}{\partial\xi^{\epsilon}} + A_2, \tag{73}$$

In Eq. (73), A_1 ($A_1 \neq 0$) and A_2 are the constants to be determined later. ϵ is the integer that can be determined through balancing the terms H'' and H^3 in Eq. (24) by means of the homogeneous balance method as $\epsilon = 1$.

5.1. The soliton solution of Sin–Cos function

We assume that $\phi(\xi)$ has the following form[28]:

$$\phi(\xi) = ln\left(\eta_0\cos(\xi) + \eta_1 + \eta_2\sin(\xi)\right), \tag{74}$$

where η_i, $i = 0,1,2$ are constants (cannot all be zero at the same time) to be determined. Putting (74) with (73) into Eq. (24) and collecting all coefficients of $sin(\xi)$, $cos(\xi)$ and equating zero we get a algebraic system. Solving this system we obtain ;

- **Set 1:**

$$\left\{ A_1 = \frac{\sqrt{-2k^2+4kA_2-4A_2^2+4m+2\sigma}}{2}, \eta_2 = \frac{2\left(\left(-k+3A_2\right)\sigma + \left(k-2A_2\right)\left(k^2-3kA_2+4A_2^2-2m\right)\right)\eta_0}{\sqrt{-2k^2+4kA_2-4A_2^2+4m+2\sigma}\,\left(k^2-6kA_2+8A_2^2-\sigma\right)}, \eta_1 = 0 \right\}$$

where $\sigma = \sqrt{\left(k^2-4m\right)\left(k-2A_2\right)^2}$.

The soliton solution of the trigonometric function type may be obtained as follows using Eqs. (73), (74) and (20) (for conformable derivative form): See Box I where $\sigma = \sqrt{\left(k^2-4m\right)\left(k-2A_2\right)^2}$, $Y = \sqrt{-2k^2+4kA_2-4A_2^2+4m+2\sigma}$, $C = \cos\left(\frac{c\,t^{\alpha}-x\alpha}{\alpha}\right)$, $S = \sin\left(\frac{c\,t^{\alpha}-x\alpha}{\alpha}\right)$.

- **Set 2:**

$$\left\{ A_1 = \frac{\sqrt{-2k^2+2\sqrt{k^4-4\,m\,k^2}+4m}}{2}, A_2 = 0, \eta_0 = \frac{2\left(-\frac{k^2}{2}+\frac{\sqrt{k^4-4k^2m}}{2}\right)\eta_2}{k2A_1}, \eta_1 = 0 \right\}$$

Considering these values, another soliton solution of the trigonometric function type may be obtained as follows using Eqs. (73), (74) and (20) (for conformable derivative form):

$$\begin{aligned} u_{22} &= -\frac{2A_1\left(2CA_1k - S\left(k^2-\sqrt{k^2\left(k^2-4m\right)}\right)\right)}{4SA_1k + 2C\left(k^2-\sqrt{k^2\left(k^2-4m\right)}\right)} \\ v_{22} &= -\frac{2\left(2CA_1k - S\left(k^2-\sqrt{k^2\left(k^2-4m\right)}\right)\right)A_1}{\sqrt{\delta}\left(4SA_1k + 2C\left(k^2-\sqrt{k^2\left(k^2-4m\right)}\right)\right)} \end{aligned} \tag{77}$$

where $C = \cos\left(\frac{c\,t^{\alpha}-x\alpha}{\alpha}\right)$, $S = \sin\left(\frac{c\,t^{\alpha}-x\alpha}{\alpha}\right)$.

- **Set 3:**

$$\left\{ A_1 = \frac{\sqrt{-2k^2+2\sqrt{k^4-4\,m\,k^2}+4m}}{4}, A_2 = \frac{k\left(-k^2+3m+\sqrt{k^4-4k^2m}\right)}{-k^2+3\sqrt{k^4-4k^2m}}, \eta_0 = \mathrm{i}\eta_2, \eta_1 = 0 \right\}.$$

Keeping these values in mind, another soliton solution of the trigonometric function type may be obtained as follows using Eqs. (73), (74) and (20) (for conformable derivative form):

$$\begin{aligned} u_{23}(x,t) &= \frac{A_1\left(\mathrm{i}S+C\right)}{4\,\mathrm{i}C-4S} + A_2 \\ v_{23}(x,t) &= \frac{-\frac{A_1(S\mathrm{i}+C)}{4(-\mathrm{i}C+S)} + A_2}{\sqrt{\delta}} \end{aligned} \tag{78}$$

where $C = \cos\left(\frac{c\,t^{\alpha}-x\alpha}{\alpha}\right)$, $S = \sin\left(\frac{c\,t^{\alpha}-x\alpha}{\alpha}\right)$.

- **Set 4:**

$$\left\{ A_1 = \frac{\sqrt{-k^2+4m}}{2}, A_2 = k/2, \eta_0 = i\eta_2, \eta_1 = 0 \right\}.$$

Using Eqs. (73), (74), and (20), another soliton solution of the trigonometric function type may be produced by taking these values into consideration. In the case of the conformable derivative form:

$$\begin{aligned} u_{24}(x,t) &= \frac{\sqrt{-k^2+4m}\left(\mathrm{i}\sin\left(\frac{c\,t^{\alpha}-x\alpha}{\alpha}\right)+\cos\left(\frac{c\,t^{\alpha}-x\alpha}{\alpha}\right)\right)}{2\,\mathrm{i}\cos\left(\frac{c\,t^{\alpha}-x\alpha}{\alpha}\right) - 2\sin\left(\frac{c\,t^{\alpha}-x\alpha}{\alpha}\right)} + \frac{k}{2} \\ v_{24}(x,t) &= \frac{\frac{\sqrt{-k^2+4m}\left(\mathrm{I}\sin\left(\frac{c\,t^{\alpha}-x\alpha}{\alpha}\right)+\cos\left(\frac{c\,t^{\alpha}-x\alpha}{\alpha}\right)\right)}{\mathrm{I}\cos\left(\frac{c\,t^{\alpha}-x\alpha}{\alpha}\right)-\sin\left(\frac{c\,t^{\alpha}-x\alpha}{\alpha}\right)} + k}{2\sqrt{\delta}}. \end{aligned} \tag{79}$$

5.2. The soliton solution of rational function

The following expression of can also be assumed[28]:

$$\phi(\xi) = ln(\eta_1\xi + \eta_0), \tag{80}$$

where η_i, $i = 0,1$ are constants is the constants to be determined later. Substituting Eqs. (73) and (80) into Eq. (24), and taking the appropriate

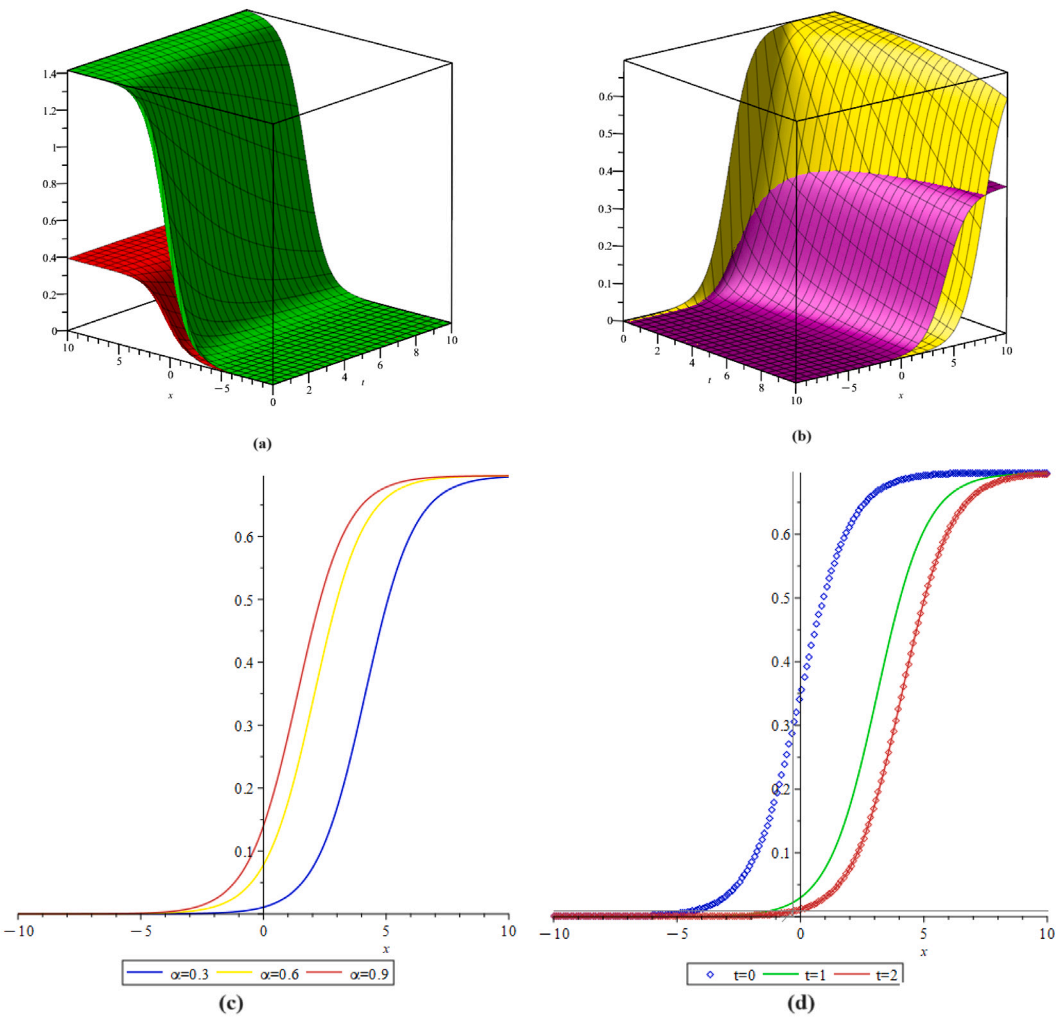

Fig. 1. **(a)** 3D wave profile of u_1 (green), v_1 (red) given in (27) for conformable-derivative with $\alpha = 0.4$, $k = 2.6$. **(b)** 3D wave profile of v_1 given in (27) for conformable-derivative with $\alpha = 0.4$. Purple represents $k = 2.6$ and yellow represents $k = 3$., **(c)** 2D-plot of v_1 for different α values at $t = 1$ with $k = 3$. **(d)** 2D graph of v_1 at different moments t with $\alpha = 0.4$. From Fig. 1, it is evident that the slight increase in k has led to a notable rise in predator density. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$$u_{21} = \frac{C\left(k-2A_2\right)\left(k^2-2kA_2-2m-\sigma\right)Y - 2\left(\left(-k^2+3kA_2-2A_2^2+m\right)\sigma + \left(k-2A_2\right)^2\left(k^2-kA_2-3m\right)\right)S}{C\left(-\sigma+\left(k-2A_2\right)\left(k-4A_2\right)\right)Y - 2\left(\left(-k+3A_2\right)\sigma + \left(k-2A_2\right)\left(k^2-3kA_2+4A_2^2-2m\right)\right)S} \tag{75}$$

$$v_{21} = \frac{C\left(k-2A_2\right)\left(-k^2+2kA_2+2m+\sigma\right)Y - 2\left(\left(k^2-3kA_2+2A_2^2-m\right)\sigma - \left(k-2A_2\right)^2\left(k^2-kA_2-3m\right)\right)S}{\sqrt{\delta}\left(\left(\sigma-\left(k-2A_2\right)\left(k-4A_2\right)\right)CY - 2\left(\left(k-3A_2\right)\sigma - \left(k-2A_2\right)\left(k^2-3kA_2+4A_2^2-2m\right)\right)S\right)} \tag{76}$$

Box I.

adjustments, we can obtain an algebraic equation system, solving it, we have:

- **Set 1:**

$$\left\{ c = -\frac{\sqrt{2}\,k}{2}, m = k^2/4, A_1 = \sqrt{2}, A_2 = k/2 \right\}.$$

With respect to Eqs. (73), (80), and (20), and the previously mentioned outcomes, we may get the following algebraic soliton solutions of the rational function type for predator–prey system:

$$\begin{aligned} u_{25}(x,t) &= \frac{\eta_1\left(k^2 t^\alpha + 4\alpha\right)\sqrt{2} + 2k\alpha\left(x\eta_1+\eta_0\right)}{2\sqrt{2}\,t^\alpha k\eta_1 + 4\alpha\left(x\eta_1+\eta_0\right)} \\ v_{25}(x,t) &= \frac{\left(\eta_1\left(k^2 t^\alpha + 4\alpha\right)\sqrt{2} + 2k\alpha\left(x\eta_1+\eta_0\right)\right)\operatorname{csgn}\left(\frac{1}{k-2}\right)(k-2)}{4\sqrt{2}\,t^\alpha k\eta_1 + 8\alpha\left(x\eta_1+\eta_0\right)}. \end{aligned} \tag{81}$$

5.3. The soliton solution of sinh–cosh function

We can also assume that the form of is as follows[28]:

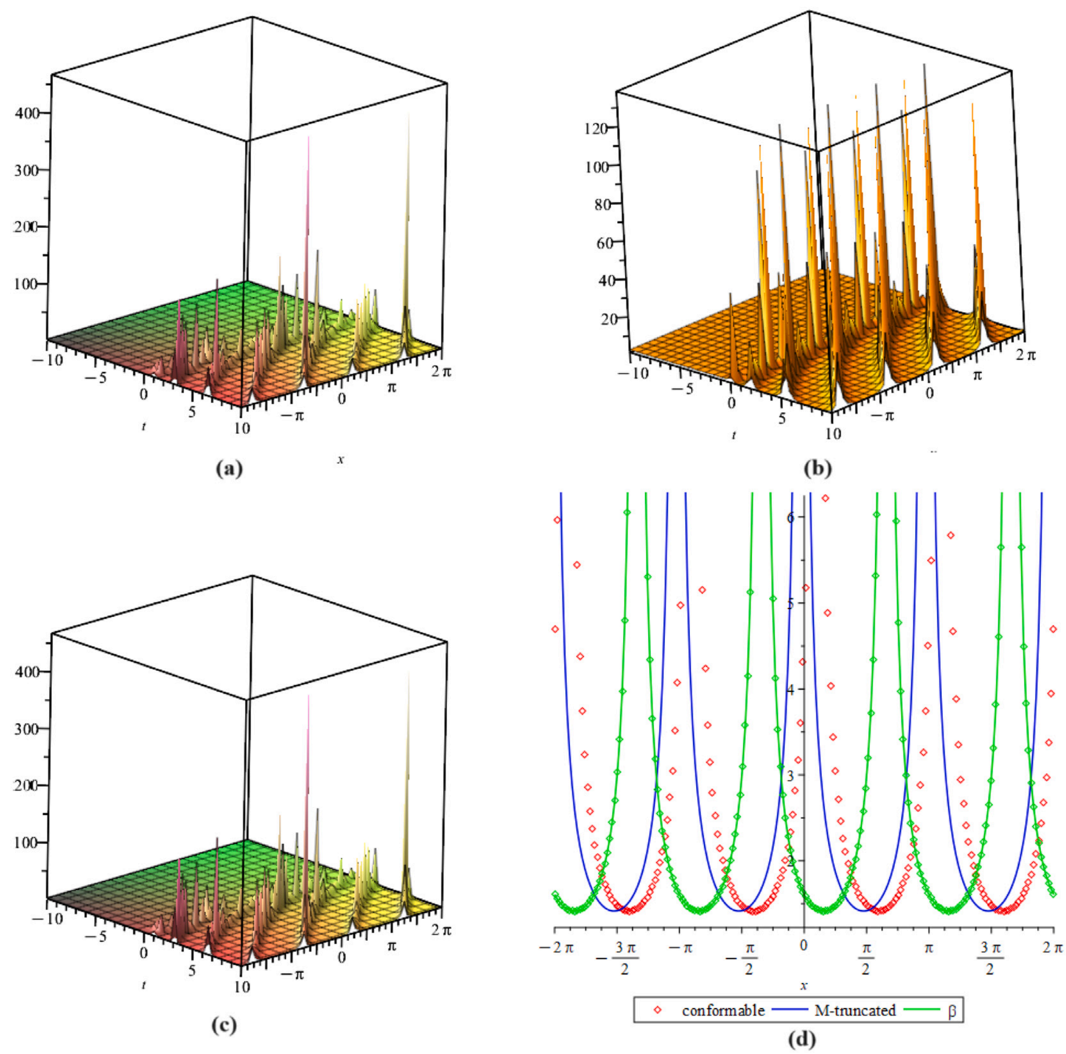

Fig. 2. **(a)** 3D wave profile of $|u_3|$ given in Eq. (32) for conformable-derivative with $\alpha = 0.5$ **(b)** 3D wave profile of $|u_3|$ given for Beta-derivative with $\alpha = 0.5$. **(c)** 3D wave profile of $|u_3|$ given for M-truncated derivative with $\alpha = 0.5$, $\beta = 1$. **(d)** 2D-plot of modulus of $|u_3|$ for considered derivatives at $t = 1$ for $\alpha = 0.5$. If you pay attention to (a), (b) and (c), it is seen that there are very similar waves for conformable and M truncated fractional derivative operators, while there are waves with higher amplitude for Beta derivatives.

$$\phi(\xi) = \ln\left(\eta_0 \cosh(\epsilon) + \eta_1 + \eta_2 \sinh(\epsilon)\right), \tag{82}$$

where η_i, $i = 0, 1, 2$ are constants (cannot all be zero at the same time) to be determined. Putting (82) with (73) into Eq. (24) and collecting all coefficients of $sin(\xi)$, $cos(\xi)$ and equating zero we get a algebraic system. Solving this system we obtain;

- **Set 1:**

$$\left\{c = -\frac{k\sqrt{2}}{2}, m = \frac{k^2}{4} - \frac{1}{2}, A_1 = \sqrt{(2)}, A_2 = \frac{k}{2} + \frac{\sqrt{2}}{2}, \eta_2 = -\eta_0\right\}.$$

With respect to Eqs. (73), (82), and (20), and the previously mentioned outcomes, we may get the following algebraic soliton solutions of the rational function type for system (16):

$$u_{26}(x,t) = \frac{-\eta_0\left(k-\sqrt{2}\right)\cosh\left(\frac{k\sqrt{2}\,t^\alpha+2x\alpha}{2\alpha}\right) + \eta_0\left(k-\sqrt{2}\right)\sinh\left(\frac{k\sqrt{2}\,t^\alpha+2x\alpha}{2\alpha}\right) - \eta_1\left(k+\sqrt{2}\right)}{2\eta_0\sinh\left(\frac{k\sqrt{2}\,t^\alpha+2x\alpha}{2\alpha}\right) - 2\eta_0\cosh\left(\frac{k\sqrt{2}\,t^\alpha+2x\alpha}{2\alpha}\right) - 2\eta_1}$$

$$v_{26}(x,t) = \frac{\dfrac{\sqrt{2}\left(\eta_0\sinh\left(x+\frac{k\sqrt{2}\,t^\alpha}{2\alpha}\right) - \eta_0\cosh\left(x+\frac{k\sqrt{2}\,t^\alpha}{2\alpha}\right)\right)}{\eta_0\cosh\left(x+\frac{k\sqrt{2}\,t^\alpha}{2\alpha}\right) + \eta_1 - \eta_0\sinh\left(x+\frac{k\sqrt{2}\,t^\alpha}{2\alpha}\right)} + \dfrac{k}{2} + \dfrac{\sqrt{2}}{2}}{\sqrt{\delta}}. \tag{83}$$

- **Set 2:**

$$\left\{c = -\frac{\sqrt{2}\,k}{2}, m = \frac{k^2}{4} - 2, A_1 = \sqrt{2}, A_2 = k/2, \eta_1 = 0, \eta_2 = 0\right\}$$

and so by following the steps above we obtain:

$$u_{27}(x,t) = \sqrt{2}\tanh\left(\frac{\sqrt{2}\,k\,t^\alpha + 2x\alpha}{2\alpha}\right) + \frac{k}{2}$$

$$v_{27}(x,t) = \frac{2\sqrt{2}\tanh\left(\frac{\sqrt{2}\,k\,t^\alpha+2x\alpha}{2\alpha}\right) + k}{2\sqrt{\delta}}. \tag{84}$$

- **Set 3:**

$$\left\{k = \frac{(c+3)\sqrt{2}}{2}, m = c+1, A_1 = \sqrt{(2)}, A_2 = 0, \eta_2 = \eta_0\right\}$$

with help of Eqs. (73), (82), and (20), and Set 3 we obtain:

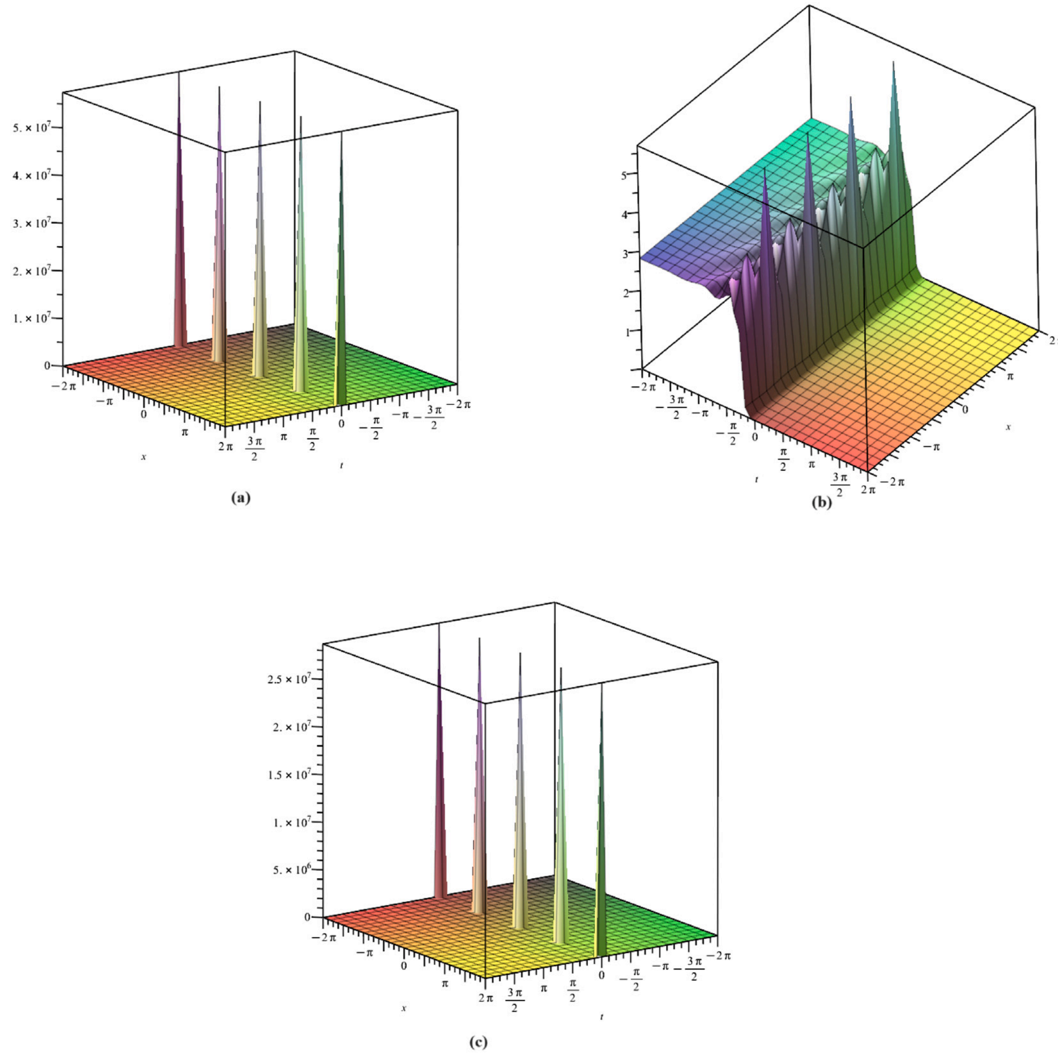

Fig. 3. **(a)** 3D wave profile of u_{13} for conformable-derivative with $\alpha = 0.6$, $m = 0.9$. **(b)** 3D wave profile of $u_1 3$ corresponding to the Beta fractional derivative with $\alpha = 0.6$, $m = 0.9$. **(c)** 3D wave profile of $u_1 3$ corresponding to the M-truncated fractional derivative with $\alpha = 0.6$, $m = 0.9$, $\beta = 1$. From Fig. 3 we can deduce that the Beta fractional derivative operator produced kink-breather solitons and had considerably distinct behaviour from the conformable and M truncated fractional derivative operators.

$$u_{28}(x,t) = \frac{\sqrt{2}\,\eta_0\left(\sinh\left(\frac{c\,t^{\alpha}-x\alpha}{\alpha}\right) - \cosh\left(\frac{c\,t^{\alpha}-x\alpha}{\alpha}\right)\right)}{\eta_0 \sinh\left(\frac{c\,t^{\alpha}-x\alpha}{\alpha}\right) - \eta_0 \cosh\left(\frac{c\,t^{\alpha}-x\alpha}{\alpha}\right) - \eta_1}$$

$$v_{28}(x,t) = \frac{\left(\sinh\left(\frac{c\,t^{\alpha}-x\alpha}{\alpha}\right) - \cosh\left(\frac{c\,t^{\alpha}-x\alpha}{\alpha}\right)\right)\eta_0}{\sqrt{-\frac{1}{(c+3)\sqrt{2}-2c-4}}\left(\eta_0\left(\sinh\left(\frac{c\,t^{\alpha}-x\alpha}{\alpha}\right) - \cosh\left(\frac{c\,t^{\alpha}-x\alpha}{\alpha}\right)\right) - \eta_1\right)}. \tag{85}$$

6. Graphical representation and discussions

In this section, we will give various graphic simulations of the obtained solutions and physically interpret the behaviour of the solutions.

We consider the 3D and 2D graphics of the solutions given in Fig. 1. Here, conformable, Beta and M-truncated derivative operators show the same behaviour. In (a), kink type waves are observed for u_1 and v_1. We know that the prey and predator densities at location x and time t are represented by u and v respectively. If we pay attention here, we can conclude that the density of the prey increases faster than the density of the predator. In accordance with observation in nature, we see that as the density of prey increases, the density of predator also increases. We saw in (b) how the predator density was impacted by a little increase in k. It is well known that k represents the rise in the birth rate brought on by prey consumption. As a result, it is evident that the slight increase in k has led to a notable rise in predator density. The behaviour of v_1 in (c) was noted for 0.3, 0.6, and 0.9 values of α; no notable shift in behaviour was seen for different values of α. In (d), the change of v_1 over time was examined. In Fig. 2 (a), (b) and (c), we observed the behaviour of the solution with 3D plots for the conformable, Beta and M-truncated fractional derivative operators, respectively. On the other hand in (d), we showed the 2D plot of the solution at time $t = 1$ in the same graph for different derivative operators. Although we observe periodic-like waves for each derivative operator, we clearly see that there are very similar waves for conformable and M truncated fractional

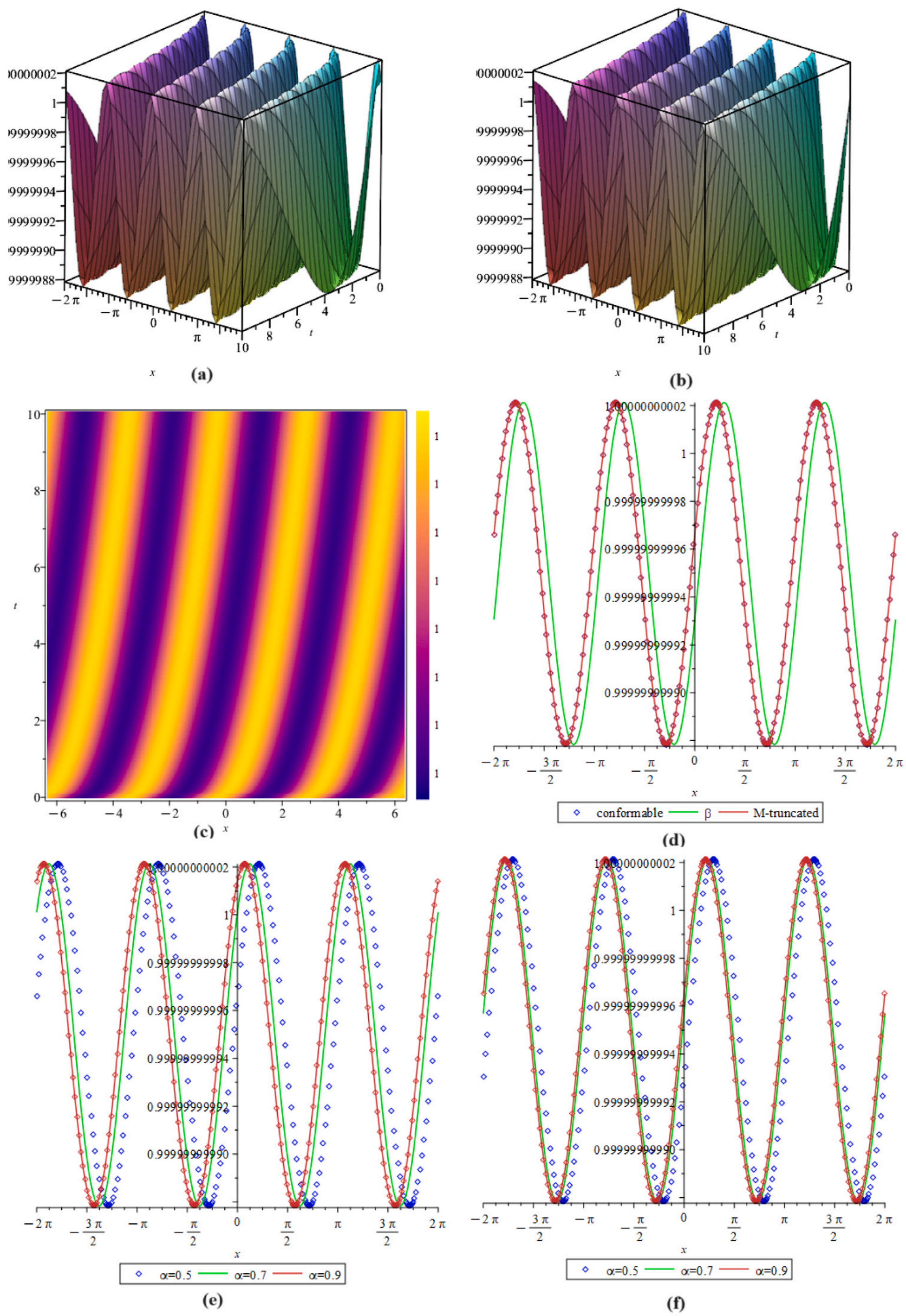

Fig. 4. **(a)** 3D periodic wave profile of $|u_{21}|$, given in (75) for conformable-derivative and also for M-truncated derivative (3D plots of its exactly same for $\beta = 1$) with $\alpha = 0.5$, $m = 1$, $k = 0.5$, $\eta_0 = 0.5$, $c = 0.5$, $A_2 = 1$. **(b)** 3D wave profile of $|u_{21}|$ corresponding to the Beta fractional derivative with $\alpha = 0.5$, $m = 1$, $k = 0.5$, $\eta_0 = 0.5$, $c = 0.5$, $A_2 = 1$. **(c)** Density plot of u_{21} given in (75) with $\alpha = 0.5$, $m = 1$, $k = 0.5$, $\eta_0 = 0.5$, $c = 0.5$, $A_2 = 1$. **(d)** 2D-plot of modulus of $|u_{21}|$ for considered derivatives at $t = 1$ for $\alpha = 0.5$.**(e)** 2D-plot of modulus of u_{21} for conformable and M-truncated derivatives at $t = 1$ for α values written at the bottom of each chart.**(f)** 2D-plot of modulus of u_{21} for Beta derivative at $t = 1$ for α values written at the bottom of each chart. In the case of changing α values, the Beta derivative operator is more stable when (f) and (e) are considered jointly.

derivative operators, while there are waves with higher amplitude for Beta derivatives. In Fig. 3:, we observed the effect of conformable, Beta and M-truncated fractional derivative operators on the solutions with 3D plots. In Fig. 3, unlike other solutions, singular solitons are present in the plots made for M-truncated and conformable fractional derivative operators and behaviour of the two operators is identical. Furthermore, the Beta fractional derivative operator produced kink-breather solitons and had considerably distinct behaviour from the conformable and M truncated fractional derivative operators.

In Fig. 4 (a), the 3D plot of the u_{21} solution for the conformable and M-truncated fractional derivative operators is presented, and in (b), the 3D plot of the u_{21} solution for the Beta derivative operator is discussed. Periodic waves are observed for all derivative operators, but if you pay attention, there are small differences for the Beta derivative operator. This situation is clearly shown by comparing the derivative operators for $\alpha = 0.5$ in (d). We also noted, using Fig. 4, how changing the values of α affects the solutions. This analysis was done for the Beta derivative operator in (f) and for the conformable and M-truncated derivative operators in (e). When α values in the solutions corresponding to the fractional derivative operators under consideration changed, no notable changes were noticed. In the case of changing α values, the Beta derivative operator is more stable when (f) and (e) are considered jointly.

7. Conclusion

In this study, we first discussed the predator prey system with different fractional derivative operators. We analysed the solutions of the model using two different new techniques. In addition, we presented various graphical representations of the various types of solutions obtained (kink, periodic, singular, kink-breather). Another important contribution of our study to the literature is the comparison of the effect of different fractional derivative operators on the solutions. According to our observations, the conformable and M truncated fractional derivative operators always exhibited very similar behaviour, while the Beta fractional derivative operator behaved differently from other derivative operators from time to time. On the other hand, we examined the effect of some parameters on the solutions. We believe that the solutions we obtained are different from those in the literature.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

This study was supported by the Scientific Research Project Unit (BAP) of Bursa Uludağ University, under project number: FHIZ-2023-1680. The authors thank Bursa Uludağ University.

Appendix

The forms of the obtained solutions corresponding to the M-truncated derivative operator are given below.

$$u_1(x,t) = 3\sqrt{2} + \frac{\sqrt{2}\left(-2e^{x-\frac{\left(k\sqrt{2}-3\right)t^{\alpha}\Gamma(\beta+1)}{\alpha}}-3\right)}{e^{x-\frac{\left(k\sqrt{2}-3\right)t^{\alpha}\Gamma(\beta+1)}{\alpha}}+1}$$

$$v_1(x,t) = \frac{3\sqrt{2}+\frac{\sqrt{2}\left(-2e^{x-\frac{\left(k\sqrt{2}-3\right)t^{\alpha}\Gamma(\beta+1)}{\alpha}}-3\right)}{e^{x-\frac{\left(k\sqrt{2}-3\right)t^{\alpha}\Gamma(\beta+1)}{\alpha}}+1}}{\sqrt{\frac{1}{k\sqrt{2}-1-k}}}$$

$$u_2(x,t) = A_0 + \frac{6\sqrt{2}\left(e^{x-\frac{\left(A_0\sqrt{2}-5\right)t^{\alpha}\Gamma(\beta+1)}{\alpha}}+1\right)}{-2e^{x-\frac{\left(A_0\sqrt{2}-5\right)t^{\alpha}\Gamma(\beta+1)}{\alpha}}-3}$$

$$v_2(x,t) = \frac{A_0+\frac{6\sqrt{2}\left(e^{x-\frac{\left(A_0\sqrt{2}-5\right)t^{\alpha}\Gamma(\beta+1)}{\alpha}}+1\right)}{-2e^{x-\frac{\left(A_0\sqrt{2}-5\right)t^{\alpha}\Gamma(\beta+1)}{\alpha}}-3}}{\sqrt{\frac{1}{A_0^2-5A_0\sqrt{2}+13-2A_0+5\sqrt{2}}}}$$

$$u_3(x,t) = \frac{\sqrt{2}\left((1+\mathrm{I})\cos\left(\frac{c\,t^{\alpha}\Gamma(\beta+1)-x\alpha}{\alpha}\right)+(-1+\mathrm{I})\sin\left(\frac{c\,t^{\alpha}\Gamma(\beta+1)-x\alpha}{\alpha}\right)\right)}{\cos\left(\frac{c\,t^{\alpha}\Gamma(\beta+1)-x\alpha}{\alpha}\right)+\sin\left(\frac{c\,t^{\alpha}\Gamma(\beta+1)-x\alpha}{\alpha}\right)}$$

$$v_3(x,t) = \frac{\sqrt{2}\left((1+\mathrm{I})\cos\left(\frac{c\,t^{\alpha}\Gamma(\beta+1)-x\alpha}{\alpha}\right)+(-1+\mathrm{I})\sin\left(\frac{c\,t^{\alpha}\Gamma(\beta+1)-x\alpha}{\alpha}\right)\right)}{\sqrt{-\frac{1}{3-4\mathrm{I}+(1+3\mathrm{I})\sqrt{2}}}\left(\cos\left(\frac{c\,t^{\alpha}\Gamma(\beta+1)-x\alpha}{\alpha}\right)+\sin\left(\frac{c\,t^{\alpha}\Gamma(\beta+1)-x\alpha}{\alpha}\right)\right)}$$

$$u_4(x,t) = -\frac{2\sqrt{2}\sin\left(\frac{x\alpha+2t^{\alpha}\Gamma(\beta+1)}{\alpha}\right)}{\cos\left(\frac{x\alpha+2t^{\alpha}\Gamma(\beta+1)}{\alpha}\right)-\sin\left(\frac{x\alpha+2t^{\alpha}\Gamma(\beta+1)}{\alpha}\right)}$$

$$v_4(x,t) = -\frac{2\sqrt{2}\sqrt{5-2\sqrt{2}}\sin\left(\frac{x\alpha+2t^{\alpha}\Gamma(\beta+1)}{\alpha}\right)}{\cos\left(\frac{x\alpha+2t^{\alpha}\Gamma(\beta+1)}{\alpha}\right)-\sin\left(\frac{x\alpha+2t^{\alpha}\Gamma(\beta+1)}{\alpha}\right)}$$

$$u_5(x,t) = \frac{3\sqrt{2}}{2} + \frac{5\sqrt{2}\sin\left(x-\frac{7t^{\alpha}\Gamma(\beta+1)}{\alpha}\right)}{\cos\left(x-\frac{7t^{\alpha}\Gamma(\beta+1)}{\alpha}\right)+2\sin\left(x-\frac{7t^{\alpha}\Gamma(\beta+1)}{\alpha}\right)}$$

$$v_5(x,t) = \frac{\sqrt{110-28\sqrt{2}}\sqrt{2}\left(3\cos\left(\frac{7t^{\alpha}\Gamma(\beta+1)-x\alpha}{\alpha}\right)-16\sin\left(\frac{7t^{\alpha}\Gamma(\beta+1)-x\alpha}{\alpha}\right)\right)}{4\cos\left(\frac{7t^{\alpha}\Gamma(\beta+1)-x\alpha}{\alpha}\right)-8\sin\left(\frac{7t^{\alpha}\Gamma(\beta+1)-x\alpha}{\alpha}\right)}$$

$$u_6(x,t) = 2\sqrt{2} + \mathrm{I}\sqrt{2} + \frac{\left(\frac{\left(2\sqrt{2}+\mathrm{I}\sqrt{2}\right)^3}{8}-\frac{11\sqrt{2}}{2}-\frac{11\mathrm{I}\sqrt{2}}{4}\right)\sin\left(x-\frac{7t^{\alpha}\Gamma(\beta+1)}{\alpha}\right)}{\cos\left(x-\frac{7t^{\alpha}\Gamma(\beta+1)}{\alpha}\right)+2\sin\left(x-\frac{7t^{\alpha}\Gamma(\beta+1)}{\alpha}\right)}$$

$$v_6(x,t) = \frac{2\left(-\left(1+\frac{\mathrm{I}}{2}\right)\cos\left(\frac{7t^{\alpha}\Gamma(\beta+1)-x\alpha}{\alpha}\right)+\left(-\frac{1}{2}+\mathrm{I}\right)\sin\left(\frac{7t^{\alpha}\Gamma(\beta+1)-x\alpha}{\alpha}\right)\right)}{\sqrt{-\frac{1}{6-28\mathrm{I}+(7+6\mathrm{I})\sqrt{2}}}\left(-\cos\left(\frac{7t^{\alpha}\Gamma(\beta+1)-x\alpha}{\alpha}\right)+2\sin\left(\frac{7t^{\alpha}\Gamma(\beta+1)-x\alpha}{\alpha}\right)\right)}$$

$$u_7(x,t) = \frac{5\sqrt{2}}{2} - \frac{3\sqrt{2}\sinh\left(x+\frac{9t^{\alpha}\Gamma(\beta+1)}{\alpha}\right)}{\cosh\left(x+\frac{9t^{\alpha}\Gamma(\beta+1)}{\alpha}\right)-2\sinh\left(x+\frac{9t^{\alpha}\Gamma(\beta+1)}{\alpha}\right)}$$

$$v_7(x,t) = \left(\frac{5\sqrt{2}}{2} - \frac{3\sqrt{2}\sinh\left(x+\frac{9t^{\alpha}\Gamma(\beta+1)}{\alpha}\right)}{\cosh\left(x+\frac{9t^{\alpha}\Gamma(\beta+1)}{\alpha}\right)-2\sinh\left(x+\frac{9t^{\alpha}\Gamma(\beta+1)}{\alpha}\right)}\right)\sqrt{\frac{79}{2}-9\sqrt{2}}$$

$$u_8(x,t) = 3\sqrt{2} + \frac{3\sqrt{2}\sinh\left(x+\frac{9t^{\alpha}\Gamma(\beta+1)}{\alpha}\right)}{\cosh\left(x+\frac{9t^{\alpha}\Gamma(\beta+1)}{\alpha}\right)-2\sinh\left(x+\frac{9t^{\alpha}\Gamma(\beta+1)}{\alpha}\right)}$$

$$v_8(x,t) = \left(3\sqrt{2} + \frac{3\sqrt{2}\sinh\left(x+\frac{9t^{\alpha}\Gamma(\beta+1)}{\alpha}\right)}{\cosh\left(x+\frac{9t^{\alpha}\Gamma(\beta+1)}{\alpha}\right)-2\sinh\left(x+\frac{9t^{\alpha}\Gamma(\beta+1)}{\alpha}\right)}\right)\sqrt{23-\frac{15\sqrt{2}}{2}}$$

$$u_9(x,t) = \sqrt{2} - \frac{\sqrt{2}}{e^{x-\frac{\left(k\sqrt{2}-3\right)t^{\alpha}\Gamma(\beta+1)}{\alpha}}+1}$$

$$v_9(x,t) = \frac{\sqrt{2}\,e^{\frac{\left(-k\sqrt{2}+3\right)t^{\alpha}\Gamma(\beta+1)+x\alpha}{\alpha}}}{\left(e^{\frac{\left(-k\sqrt{2}+3\right)t^{\alpha}\Gamma(\beta+1)+x\alpha}{\alpha}}+1\right)\sqrt{\frac{1}{k\sqrt{2}-1-k}}}$$

$$u_{10}(x,t)=A_0-\frac{\sqrt{2}}{e^{x+\frac{\sqrt{2}\left(-\sqrt{2}+2A_0\right)t^{\alpha}\Gamma(\beta+1)}{2\alpha}}+1}$$

$$v_{10}=\frac{A_0 e^{\frac{t^{\alpha}\left(\sqrt{2}A_0-1\right)\Gamma(\beta+1)+x\alpha}{\alpha}}-\sqrt{2}+A_0}{\left(e^{\frac{t^{\alpha}\left(\sqrt{2}A_0-1\right)\Gamma(\beta+1)+x\alpha}{\alpha}}+1\right)\sqrt{\frac{1}{\left(A_0-1\right)\left(A_0-\sqrt{2}-1\right)}}}$$

$$u_{11}(x,t)=2\sqrt{2}+\frac{\sqrt{2}\left(-e^{x-\frac{\left(\sqrt{2}k-3\right)t^{\alpha}\Gamma(\beta+1)}{\alpha}}-2\right)}{e^{x-\frac{\left(\sqrt{2}k-3\right)t^{\alpha}\Gamma(\beta+1)}{\alpha}}+1}$$

$$v_{11}(x,t)=\frac{\sqrt{2}\,e^{\frac{\left(-\sqrt{2}k+3\right)t^{\alpha}\Gamma(\beta+1)+x\alpha}{\alpha}}}{\left(e^{\frac{\left(-\sqrt{2}k+3\right)t^{\alpha}\Gamma(\beta+1)+x\alpha}{\alpha}}+1\right)\sqrt{\frac{1}{\sqrt{2}k-1-k}}}$$

$$u_{12}(x,t)=\frac{\left(A_0-2\sqrt{2}\right)e^{\frac{\left(-A_0\sqrt{2}+3\right)t^{\alpha}\Gamma(\beta+1)+x\alpha}{\alpha}}+2A_0-2\sqrt{2}}{e^{\frac{\left(-A_0\sqrt{2}+3\right)t^{\alpha}\Gamma(\beta+1)+x\alpha}{\alpha}}+2}$$

$$v_{12}(x,t)=\frac{\left(A_0-2\sqrt{2}\right)e^{\frac{\left(-A_0\sqrt{2}+3\right)t^{\alpha}\Gamma(\beta+1)+x\alpha}{\alpha}}+2A_0-2\sqrt{2}}{\sqrt{\frac{1}{\left(-3A_0+3\right)\sqrt{2}+A_0^2-2A_0+5}}\left(e^{\frac{\left(-A_0\sqrt{2}+3\right)t^{\alpha}\Gamma(\beta+1)+x\alpha}{\alpha}}+2\right)}$$

$$u_{13}(x,t)=\mathrm{I}\sqrt{2}-\frac{\sqrt{2}\cos\left(x-\frac{\mathrm{I}(m+4)t^{\alpha}\Gamma(\beta+1)}{2\alpha}\right)}{\sin\left(x-\frac{\mathrm{I}(m+4)t^{\alpha}\Gamma(\beta+1)}{2\alpha}\right)}$$

$$v_{13}(x,t)=\frac{\sqrt{2}\left(\mathrm{I}+\cot\left(\frac{\mathrm{I}t^{\alpha}(m+4)\Gamma(\beta+1)-2x\alpha}{2\alpha}\right)\right)}{2\sqrt{\frac{1}{4+\mathrm{I}\sqrt{2}(m-8)+4m}}}$$

$$u_{14}(x,t)=\sqrt{m-2}+\sqrt{2}\cot\left(\frac{\sqrt{2}\sqrt{m-2}\,t^{\alpha}\Gamma(\beta+1)-x\alpha}{\alpha}\right)$$

$$v_{14}(x,t)=\frac{\sqrt{m-2}+\sqrt{2}\cot\left(\frac{\sqrt{2}\sqrt{m-2}\,t^{\alpha}\Gamma(\beta+1)-x\alpha}{\alpha}\right)}{\sqrt{\frac{1}{m-2\sqrt{m-2}+1}}}$$

$$u_{15}(x,t)=\sqrt{2}\left(2\mathrm{I}-\tan\left(\frac{\mathrm{I}(m+16)t^{\alpha}\Gamma(\beta+1)+4x\alpha}{4\alpha}\right)+\cot\left(\frac{\mathrm{I}(m+16)t^{\alpha}\Gamma(\beta+1)+4x\alpha}{4\alpha}\right)\right)$$

$$v_{15}(x,t)=\frac{2\mathrm{I}-\tan\left(\frac{\mathrm{I}(m+16)t^{\alpha}\Gamma(\beta+1)+4x\alpha}{4\alpha}\right)+\cot\left(\frac{\mathrm{I}(m+16)t^{\alpha}\Gamma(\beta+1)+4x\alpha}{4\alpha}\right)}{2\sqrt{\frac{1}{8+\mathrm{I}\sqrt{2}(m-32)+8m}}}$$

$$u_{16}(x,t)=\sqrt{2}\tan\left(\frac{\sqrt{2}\sqrt{m-8}\,t^{\alpha}\Gamma(\beta+1)+x\alpha}{\alpha}\right)+\sqrt{m-8}+\sqrt{2}\cot\left(\frac{\sqrt{2}\sqrt{m-8}\,t^{\alpha}\Gamma(\beta+1)+x\alpha}{\alpha}\right)$$

$$v_{16}(x,t)=\frac{\sqrt{m-8}-\sqrt{2}\left(\tan\left(\frac{\sqrt{2}\sqrt{m-8}\,t^{\alpha}\Gamma(\beta+1)+x\alpha}{\alpha}\right)-\cot\left(\frac{\sqrt{2}\sqrt{m-8}\,t^{\alpha}\Gamma(\beta+1)+x\alpha}{\alpha}\right)\right)}{\sqrt{\frac{1}{m-2\sqrt{m-8}+1}}}$$

$$u_{17}(x,t)=\sqrt{2}\left(-1+\tanh\left(\frac{(m-4)t^{\alpha}\Gamma(\beta+1)+2x\alpha}{2\alpha}\right)\right)$$

$$v_{17}(x,t)=\frac{\sqrt{2}\left(-1+\tanh\left(\frac{(m-4)t^{\alpha}\Gamma(\beta+1)+2x\alpha}{2\alpha}\right)\right)}{2\sqrt{-\frac{1}{\sqrt{2}(m+8)-4m-4}}}$$

$$u_{18}(x,t)=\sqrt{m+2}+\sqrt{2}\coth\left(\frac{\sqrt{2}\sqrt{m+2}\,t^{\alpha}\Gamma(\beta+1)-x\alpha}{\alpha}\right)$$

$$v_{18}(x,t)=\frac{\sqrt{m+2}+\sqrt{2}\coth\left(\frac{\sqrt{2}\sqrt{m+2}\,t^{\alpha}\Gamma(\beta+1)-x\alpha}{\alpha}\right)}{\sqrt{\frac{1}{m-2\sqrt{m+2}+1}}}$$

$$u_{19}(x,t)=\sqrt{2}\left(\coth\left(\frac{(m-16)t^{\alpha}\Gamma(\beta+1)+4x\alpha}{4\alpha}\right)-2+\tanh\left(\frac{(m-16)t^{\alpha}\Gamma(\beta+1)+4x\alpha}{4\alpha}\right)\right)$$

$$v_{19}(x,t)=\frac{\coth\left(\frac{(m-16)t^{\alpha}\Gamma(\beta+1)+4x\alpha}{4\alpha}\right)-2+\tanh\left(\frac{(m-16)t^{\alpha}\Gamma(\beta+1)+4x\alpha}{4\alpha}\right)}{2\sqrt{-\frac{1}{\sqrt{2}(m+32)-8m-8}}}$$

$$u_{20}(x,t)=\sqrt{2}\coth\left(\frac{\sqrt{2}\sqrt{m+8}\,t^{\alpha}\Gamma(\beta+1)-x\alpha}{\alpha}\right)+\sqrt{2}\tanh\left(\frac{\sqrt{2}\sqrt{m+8}\,t^{\alpha}\Gamma(\beta+1)-x\alpha}{\alpha}\right)+\sqrt{m+8}$$

$$v_{20}(x,t)=\frac{\sqrt{m+8}+\left(\coth\left(\frac{\sqrt{2}\sqrt{m+8}\,t^{\alpha}\Gamma(\beta+1)-x\alpha}{\alpha}\right)+\tanh\left(\frac{\sqrt{2}\sqrt{m+8}\,t^{\alpha}\Gamma(\beta+1)-x\alpha}{\alpha}\right)\right)\sqrt{2}}{\sqrt{\frac{1}{m-2\sqrt{m+8}+1}}}$$

$$u_{21}(x,t)=\frac{C(k-2A_2)(-k^2+2kA_2+2m+\sigma)Y-2S\left(\left(k^2-3kA_2+2A_2^2-m\right)\sigma-(k-2A_2)^2\left(k^2-kA_2-3m\right)\right)}{(\sigma-(k-2A_2)(k-4A_2))CY-2S\left((k-3A_2)\sigma-(k-2A_2)\left(k^2-3kA_2+4A_2^2-2m\right)\right)}$$

$$v_{21}(x,t)=\frac{C(k-2A_2)(-k^2+2kA_2+2m+\sigma)Y-2S\left(\left(k^2-3kA_2+2A_2^2-m\right)\sigma-(k-2A_2)^2\left(k^2-kA_2-3m\right)\right)}{\sqrt{-\frac{1}{-m+k-1}}\left((\sigma-(k-2A_2)(k-4A_2))CY-2S\left((k-3A_2)\sigma-(k-2A_2)\left(k^2-3kA_2+4A_2^2-2m\right)\right)\right)}$$

$$u_{22}(x,t)=\frac{A1\left(C\sqrt{-2k^2+2\sqrt{k^2(k^2-4m)}+4m}\,k+S\left(-k^2+\sqrt{k^2(k^2-4m)}\right)\right)}{-S\sqrt{-2k^2+2\sqrt{k^2(k^2-4m)}+4m}\,k+C\left(-k^2+\sqrt{k^2(k^2-4m)}\right)}$$

$$v_{22}(x,t)=\frac{\left(C\sqrt{-2k^2+2\sqrt{k^2(k^2-4m)}+4m}\,k+S\left(-k^2+\sqrt{k^2(k^2-4m)}\right)\right)A_1}{\sqrt{-\frac{1}{-m+k-1}}\left(-S\sqrt{-2k^2+2\sqrt{k^2(k^2-4m)}+4m}\,k+C\left(-k^2+\sqrt{k^2(k^2-4m)}\right)\right)}$$

$$u_{23}=\frac{\sqrt{-2k^2+2\sqrt{k^4-4\,mk^2}+4m}\left(-\mathrm{I}\eta_2\sin\left(x-\frac{ct^{\alpha}\Gamma(\beta+1)}{\alpha}\right)+\eta_2\cos\left(x-\frac{ct^{\alpha}\Gamma(\beta+1)}{\alpha}\right)\right)}{4\left(\mathrm{I}\eta_2\cos\left(x-\frac{ct^{\alpha}\Gamma(\beta+1)}{\alpha}\right)+\eta_2\sin\left(x-\frac{ct^{\alpha}\Gamma(\beta+1)}{\alpha}\right)\right)}+\frac{k\left(-k^2+3m+\sqrt{k^4-4\,mk^2}\right)}{-k^2+3\sqrt{k^4-4\,mk^2}}$$

$$v_{23}=\frac{\frac{A1\left(-\mathrm{I}\eta_2\sin\left(x-\frac{ct^{\alpha}\Gamma(\beta+1)}{\alpha}\right)+\eta_2\cos\left(x-\frac{ct^{\alpha}\Gamma(\beta+1)}{\alpha}\right)\right)}{4\left(\mathrm{I}\eta_2\cos\left(x-\frac{ct^{\alpha}\Gamma(\beta+1)}{\alpha}\right)+\eta_2\sin\left(x-\frac{ct^{\alpha}\Gamma(\beta+1)}{\alpha}\right)\right)}+\frac{k\left(-k^2+3m+\sqrt{k^4-4\,mk^2}\right)}{-k^2+3\sqrt{k^4-4\,mk^2}}}{\sqrt{\frac{1}{m-k+1}}}$$

$$u_{24}=\frac{\sqrt{-k^2+4m}\left(-\mathrm{I}\eta_2\sin\left(x-\frac{ct^{\alpha}\Gamma(\beta+1)}{\alpha}\right)+\eta_2\cos\left(x-\frac{ct^{\alpha}\Gamma(\beta+1)}{\alpha}\right)\right)}{2\left(\mathrm{I}\eta_2\cos\left(x-\frac{ct^{\alpha}\Gamma(\beta+1)}{\alpha}\right)+\eta_2\sin\left(x-\frac{ct^{\alpha}\Gamma(\beta+1)}{\alpha}\right)\right)}+\frac{k}{2}$$

$$v_{24}(x,t)=\frac{\frac{\sqrt{-k^2+4m}\left(-\mathrm{I}\eta_2\sin\left(x-\frac{ct^{\alpha}\Gamma(\beta+1)}{\alpha}\right)+\eta_2\cos\left(x-\frac{ct^{\alpha}\Gamma(\beta+1)}{\alpha}\right)\right)}{2\left(\eta_2\cos\left(x-\frac{ct^{\alpha}\Gamma(\beta+1)}{\alpha}\right)\mathrm{I}+\eta_2\sin\left(x-\frac{ct^{\alpha}\Gamma(\beta+1)}{\alpha}\right)\right)}+\frac{k}{2}}{\sqrt{\frac{1}{m-k+1}}}$$

$$u_{25}(x,t)=\frac{\sqrt{2}\eta_1}{\left(x+\frac{\sqrt{2}kt^{\alpha}\Gamma(\beta+1)}{2\alpha}\right)\eta_1+\eta_0}+\frac{k}{2}$$

$$v_{25}(x,t)=\frac{\frac{\sqrt{2}\eta_1}{\left(x+\frac{\sqrt{2}kt^{\alpha}\Gamma(\beta+1)}{2\alpha}\right)\eta_1+\eta_0}+\frac{k}{2}}{\sqrt{\frac{1}{\frac{1}{4}k^2-k+1}}}$$

$$u_{26}(x,t)=\frac{\sqrt{2}\left(\eta_0\sinh\left(x+\frac{k\sqrt{2}t^{\alpha}\Gamma(\beta+1)}{2\alpha}\right)-\eta_0\cosh\left(x+\frac{k\sqrt{2}t^{\alpha}\Gamma(\beta+1)}{2\alpha}\right)\right)}{\eta_0\cosh\left(x+\frac{k\sqrt{2}t^{\alpha}\Gamma(\beta+1)}{2\alpha}\right)+\eta_1-\eta_0\sinh\left(x+\frac{k\sqrt{2}t^{\alpha}\Gamma(\beta+1)}{2\alpha}\right)}+\frac{k}{2}+\frac{\sqrt{2}}{2}$$

$$v_{26}(x,t)=\frac{\frac{\sqrt{2}\left(\eta_0\sinh\left(x+\frac{k\sqrt{2}t^{\alpha}\Gamma(\beta+1)}{2\alpha}\right)-\eta_0\cosh\left(x+\frac{k\sqrt{2}t^{\alpha}\Gamma(\beta+1)}{2\alpha}\right)\right)}{\eta_0\cosh\left(x+\frac{k\sqrt{2}t^{\alpha}\Gamma(\beta+1)}{2\alpha}\right)+\eta_1-\eta_0\sinh\left(x+\frac{k\sqrt{2}t^{\alpha}\Gamma(\beta+1)}{2\alpha}\right)}+\frac{k}{2}+\frac{\sqrt{2}}{2}}{\sqrt{\delta}}$$

$$u_{27}(x,t)=\sqrt{2}\,tanh\left(\frac{\sqrt{2}kt^{\alpha}\Gamma(\beta+1)+2x\alpha}{2\alpha}\right)+\frac{k}{2}$$

$$v_{27}(x,t)=\frac{2\sqrt{2}\,tanh\left(\frac{\sqrt{2}kt^{\alpha}\Gamma(\beta+1)+2x\alpha}{2\alpha}\right)+k}{2\sqrt{\delta}}$$

$$u_{28}(x,t)=\frac{\sqrt{2}\left(\eta_0\sinh\left(x-\frac{ct^{\alpha}\Gamma(\beta+1)}{\alpha}\right)+\eta_0\cosh\left(x-\frac{ct^{\alpha}\Gamma(\beta+1)}{\alpha}\right)\right)}{\eta_0\cosh\left(x-\frac{ct^{\alpha}\Gamma(\beta+1)}{\alpha}\right)+\eta_1+\eta_0\sinh\left(x-\frac{ct^{\alpha}\Gamma(\beta+1)}{\alpha}\right)}$$

$$v_{28}(x,t)=\frac{\sqrt{2}\left(\eta_0\sinh\left(x-\frac{ct^{\alpha}\Gamma(\beta+1)}{\alpha}\right)+\eta_0\cosh\left(x-\frac{ct^{\alpha}\Gamma(\beta+1)}{\alpha}\right)\right)}{\left(\eta_0\cosh\left(x-\frac{ct^{\alpha}\Gamma(\beta+1)}{\alpha}\right)+\eta_1+\eta_0\sinh\left(x-\frac{ct^{\alpha}\Gamma(\beta+1)}{\alpha}\right)\right)\sqrt{\delta}}$$

The forms of the obtained solutions corresponding to the Beta fractional derivative operator are given below.

$$u_1(x,t)=3\sqrt{2}+\frac{\sqrt{2}\left(-2\mathrm{e}^{x-\frac{\left(k\sqrt{2}-3\right)\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}}-3\right)}{\mathrm{e}^{x-\frac{\left(k\sqrt{2}-3\right)\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}}+1}$$

$$v_1(x,t)\frac{3\sqrt{2}+\frac{\sqrt{2}\left(-2\mathrm{e}^{x-\frac{\left(k\sqrt{2}-3\right)\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}}-3\right)}{\mathrm{e}^{x-\frac{\left(k\sqrt{2}-3\right)\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}}+1}}{\sqrt{\frac{1}{k\sqrt{2}-1-k}}}$$

$$u_2(x,t)=A_0+\frac{6\sqrt{2}\left(\mathrm{e}^{x-\frac{\left(A_0\sqrt{2}-5\right)\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}}+1\right)}{-2\mathrm{e}^{x-\frac{\left(A_0\sqrt{2}-5\right)\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}}-3}$$

$$v_2(x,t)=\frac{A_0+\frac{6\sqrt{2}\left(\mathrm{e}^{x-\frac{\left(A_0\sqrt{2}-5\right)\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}}+1\right)}{-2\mathrm{e}^{x-\frac{\left(A_0\sqrt{2}-5\right)\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}}-3}}{\sqrt{\frac{1}{A_0^2-5A_0\sqrt{2}+13-2A_0+5\sqrt{2}}}}$$

$$u_3(x,t)=\frac{\left((1+\mathrm{I})\cos\left(\frac{c\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-x\alpha}{\alpha}\right)+(-1+\mathrm{I})\sin\left(\frac{c\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-x\alpha}{\alpha}\right)\right)\sqrt{2}}{\cos\left(\frac{c\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-x\alpha}{\alpha}\right)+\sin\left(\frac{c\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-x\alpha}{\alpha}\right)}$$

$$v_3(x,t)=\frac{\left((1+\mathrm{I})\cos\left(\frac{c\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-x\alpha}{\alpha}\right)+(-1+\mathrm{I})\sin\left(\frac{c\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-x\alpha}{\alpha}\right)\right)\sqrt{2}}{\sqrt{-\frac{1}{3-4\mathrm{I}+(1+3\mathrm{I})\sqrt{2}}}\left(\cos\left(\frac{c\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-x\alpha}{\alpha}\right)+\sin\left(\frac{c\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-x\alpha}{\alpha}\right)\right)}$$

$$u_4(x,t)=\frac{2\sqrt{2}\sin\left(x+\frac{2\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)}{\cos\left(x+\frac{2\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)-\sin\left(x+\frac{2\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)}$$

$$v_4(x,t)=\frac{2\sqrt{2}\sin\left(x+\frac{2\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)\sqrt{5-2\sqrt{2}}}{\cos\left(x+\frac{2\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)-\sin\left(x+\frac{2\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)}$$

$$u_5(x,t)=\frac{\sqrt{2}\left(3\cos\left(\frac{-x\alpha+7\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)-16\sin\left(\frac{-x\alpha+7\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)\right)}{2\cos\left(\frac{-x\alpha+7\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)-4\sin\left(\frac{-x\alpha+7\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)}$$

$$v_5(x,t)=\frac{\sqrt{110-28\sqrt{2}}\,\sqrt{2}\left(3\cos\left(\frac{-x\alpha+7\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)-16\sin\left(\frac{-x\alpha+7\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)\right)}{4\cos\left(\frac{-x\alpha+7\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)-8\sin\left(\frac{-x\alpha+7\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)}$$

$$u_6(x,t)=2\sqrt{2}+\mathrm{I}\sqrt{2}+\frac{\left(\frac{\left(2\sqrt{2}+\mathrm{I}\sqrt{2}\right)^3}{8}-\frac{11\sqrt{2}}{2}-\frac{11\mathrm{I}\sqrt{2}}{4}\right)\sin\left(x-\frac{7\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)}{\cos\left(x-\frac{7\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)+2\sin\left(x-\frac{7\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)}$$

$$v_6(x,t)=\frac{(2+\mathrm{I})\cos\left(\frac{-x\alpha+7\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)+(1-2\mathrm{I})\sin\left(\frac{-x\alpha+7\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)}{\sqrt{-\frac{1}{6-28\mathrm{I}+(7+6\mathrm{I})\sqrt{2}}}\left(\cos\left(\frac{-x\alpha+7\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)-2\sin\left(\frac{-x\alpha+7\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)\right)}$$

$$u_7(x,t)=\frac{5\sqrt{2}}{2}-\frac{3\sqrt{2}\sinh\left(x+\frac{9\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)}{\cosh\left(x+\frac{9\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)-2\sinh\left(x+\frac{9\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)}$$

$$v_7(x,t)=\left(\frac{5\sqrt{2}}{2}-\frac{3\sqrt{2}\sinh\left(x+\frac{9\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)}{\cosh\left(x+\frac{9\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)-2\sinh\left(x+\frac{9\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)}\right)\sqrt{\frac{79}{2}-9\sqrt{2}}$$

$$u_8(x,t)=3\sqrt{2}+\frac{3\sqrt{2}\sinh\left(x+\frac{9\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)}{\cosh\left(x+\frac{9\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)-2\sinh\left(x+\frac{9\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)}$$

$$v_8(x,t)=\left(3\sqrt{2}+\frac{3\sqrt{2}\sinh\left(x+\frac{9\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)}{\cosh\left(x+\frac{9\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)-2\sinh\left(x+\frac{9\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)}\right)\sqrt{23-\frac{15\sqrt{2}}{2}}$$

$$u_9(x,t)=\sqrt{2}-\frac{\sqrt{2}}{\mathrm{e}^{x-\frac{\left(k\sqrt{2}-3\right)\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}}+1}$$

$$v_9(x,t)=\frac{\sqrt{2}\,\mathrm{e}^{\frac{-\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}\sqrt{2}k+x\alpha+3\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}}}{\left(\mathrm{e}^{\frac{-\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}\sqrt{2}k+x\alpha+3\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}}+1\right)\sqrt{\frac{1}{k\sqrt{2}-1-k}}}$$

$$u_{10}=A_0-\frac{\sqrt{2}}{\mathrm{e}^{x+\frac{\sqrt{2}\left(-\sqrt{2}+2A_0\right)\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{2\alpha}}+1}$$

$$v_{10}=\frac{A_0\mathrm{e}^{\frac{\sqrt{2}\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}A_0+x\alpha-\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}}-\sqrt{2}+A_0}{\left(\mathrm{e}^{\frac{\sqrt{2}\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}A_0+x\alpha-\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}}+1\right)\sqrt{\frac{1}{\left(A_0-1\right)\left(A_0-\sqrt{2}-1\right)}}}$$

$$u_{11}(x,t)=\frac{\sqrt{2}\,\mathrm{e}^{\frac{-\sqrt{2}\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}k+x\alpha+3\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}}}{\left(\mathrm{e}^{\frac{-\sqrt{2}\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}k+x\alpha+3\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}}+1\right)\sqrt{\frac{1}{\sqrt{2}k-1-k}}}$$

$$v_{11}(x,t)=\frac{\sqrt{2}\,\mathrm{e}^{\frac{-\sqrt{2}\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}k+x\alpha+3\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}}}{\left(\mathrm{e}^{\frac{-\sqrt{2}\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}k+x\alpha+3\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}}+1\right)\sqrt{\frac{1}{\sqrt{2}k-1-k}}}$$

$$u_{12}(x,t)=A_0+\frac{2\sqrt{2}\left(\mathrm{e}^{x-\frac{\left(A_0\sqrt{2}-3\right)\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}}+1\right)}{-\mathrm{e}^{x-\frac{\left(A_0\sqrt{2}-3\right)\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}}-2}$$

$$v_{12}(x,t)=\frac{\left(A_0-2\sqrt{2}\right)\mathrm{e}^{\frac{-\sqrt{2}\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}A_0+x\alpha+3\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}}+2A_0-2\sqrt{2}}{\sqrt{\frac{1}{\left(-3A_0+3\right)\sqrt{2}+A_0^2-2A_0+5}}\left(\mathrm{e}^{\frac{-\sqrt{2}\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}A_0+x\alpha+3\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}}+2\right)}$$

$$u_{13}(x,t)=\sqrt{2}\left(\mathrm{I}+\cot\left(\frac{\mathrm{I}(m+4)\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-2x\alpha}{2\alpha}\right)\right)$$

$$v_{13}(x,t)=\frac{\sqrt{2}\left(\mathrm{I}+\cot\left(\frac{\mathrm{I}(m+4)\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-2x\alpha}{2\alpha}\right)\right)}{2\sqrt{\frac{1}{4+\mathrm{I}\sqrt{2}(m-8)+4m}}}$$

$$u_{14}(x,t)=\sqrt{2}\cot\left(\frac{\sqrt{2}\sqrt{m-2}\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-x\alpha}{\alpha}\right)+\sqrt{m-2}$$

$$v_{14}(x,t)=\sqrt{2}\cot\left(\frac{\sqrt{2}\sqrt{m-2}\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-x\alpha}{\alpha}\right)+\sqrt{m-2}$$

$$u_{15}(x,t)=\sqrt{2}\left(2\mathrm{I}-\tan\left(\frac{\mathrm{I}(m+16)\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}+4x\alpha}{4\alpha}\right)+\cot\left(\frac{\mathrm{I}(m+16)\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}+4x\alpha}{4\alpha}\right)\right)$$

$$v_{15}(x,t)=\frac{2\mathrm{I}-\tan\left(\frac{\mathrm{I}(m+16)\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}+4x\alpha}{4\alpha}\right)+\cot\left(\frac{\mathrm{I}(m+16)\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}+4x\alpha}{4\alpha}\right)}{2\sqrt{\frac{1}{8+\mathrm{I}\sqrt{2}(m-32)+8m}}}$$

$$u_{16}(x,t)=\sqrt{2}\tan\left(\frac{\sqrt{2}\sqrt{m-8}\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}+x\alpha}{\alpha}\right)+\sqrt{m-8}+\sqrt{2}\cot\left(\frac{\sqrt{2}\sqrt{m-8}\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}+x\alpha}{\alpha}\right)$$

$$v_{16}(x,t)=\frac{\sqrt{m-8}+\left(\cot\left(\frac{\sqrt{2}\sqrt{m-8}\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}+x\alpha}{\alpha}\right)-\tan\left(\frac{\sqrt{2}\sqrt{m-8}\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}+x\alpha}{\alpha}\right)\right)\sqrt{2}}{\sqrt{\frac{1}{m-2\sqrt{m-8}+1}}}$$

$$u_{17}(x,t)=\sqrt{2}\left(\tanh\left(\frac{(m-4)\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}+2x\alpha}{2\alpha}\right)-1\right)$$

$$v_{17}(x,t)=\frac{\sqrt{2}\left(\tanh\left(\frac{(m-4)\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}+2x\alpha}{2\alpha}\right)-1\right)}{2\sqrt{-\frac{1}{\sqrt{2}(m+8)-4m-4}}}$$

$$u_{18}(x,t)=\sqrt{m+2}+\sqrt{2}\coth\left(\frac{\sqrt{2}\sqrt{m+2}\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-x\alpha}{\alpha}\right)$$

$$v_{18}(x,t)=\frac{\sqrt{m+2}+\sqrt{2}\coth\left(\frac{\sqrt{2}\sqrt{m+2}\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-x\alpha}{\alpha}\right)}{\sqrt{\frac{1}{m-2\sqrt{m+2}+1}}}$$

$$u_{19}(x,t)=\sqrt{2}\left(\coth\left(\frac{(m-16)\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}+4x\alpha}{4\alpha}\right)-2+\tanh\left(\frac{(m-16)\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}+4x\alpha}{4\alpha}\right)\right)$$

$$v_{19}(x,t)=-\frac{\coth\left(\frac{(m-16)\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}+4x\alpha}{4\alpha}\right)-2+\tanh\left(\frac{(m-16)\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}+4x\alpha}{4\alpha}\right)}{2\sqrt{-\frac{1}{\sqrt{2}(m+32)-8m-8}}}$$

$$u_{20}(x,t)=\sqrt{2}\coth\left(\frac{\sqrt{2}\sqrt{m+8}\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-x\alpha}{\alpha}\right)+\sqrt{2}\tanh\left(\frac{\sqrt{2}\sqrt{m+8}\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-x\alpha}{\alpha}\right)+\sqrt{m+8}$$

$$v_{20}(x,t)=\frac{\sqrt{m+8}+\left(\coth\left(\frac{\sqrt{2}\sqrt{m+8}\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-x\alpha}{\alpha}\right)+\tanh\left(\frac{\sqrt{2}\sqrt{m+8}\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-x\alpha}{\alpha}\right)\right)\sqrt{2}}{\sqrt{\frac{1}{m-2\sqrt{m+8}+1}}}$$

$$u_{21}(x,t)=\frac{C(k-2A_2)(k^2-2kA_2-2m-\sigma)Y-2S\left(\left(-k^2+3kA_2-2A_2^2+m\right)\sigma+(k-2A_2)^2\left(k^2-kA_2-3m\right)\right)}{C\left(k^2-6kA_2+8A_2^2-\sigma\right)Y-2S\left((-k+3A_2)\sigma+(k-2A_2)\left(k^2-3kA_2+4A_2^2-2m\right)\right)}$$

$$v_{21}(x,t)=\frac{C(k-2A_2)(-k^2+2kA_2+2m+\sigma)Y-2S\left(\left(k^2-3kA_2+2A_2^2-m\right)\sigma-(k-2A_2)^2\left(k^2-kA_2-3m\right)\right)}{\sqrt{-\frac{1}{-m+k-1}}\left(C\left(-k^2+6kA_2-8A_2^2+\sigma\right)Y-2S\left((k-3A_2)\sigma-(k-2A_2)\left(k^2-3kA_2+4A_2^2-2m\right)\right)\right)}$$

$$u_{22}(x,t)=-\frac{A1\left(C\sqrt{-2k^2+2\sqrt{k^2(k^2-4m)}+4mk}-S\left(k^2-\sqrt{k^2(k^2-4m)}\right)\right)}{S\sqrt{-2k^2+2\sqrt{k^2(k^2-4m)}+4mk}+C\left(k^2-\sqrt{k^2(k^2-4m)}\right)}$$

$$v_{22}(x,t)=\frac{\left(C\sqrt{-2k^2+2\sqrt{k^2(k^2-4m)}+4mk}-S\left(k^2-\sqrt{k^2(k^2-4m)}\right)\right)A1}{\sqrt{-\frac{1}{-m+k-1}}\left(S\sqrt{-2k^2+2\sqrt{k^2(k^2-4m)}+4mk}+C\left(k^2-\sqrt{k^2(k^2-4m)}\right)\right)}$$

$$u_{23}(x,t)=\frac{A1\left(-\mathrm{I}\eta_2\sin\left(x-\frac{c\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)+\eta_2\cos\left(x-\frac{c\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)\right)}{4\left(\mathrm{I}\eta_2\cos\left(x-\frac{c\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)+\eta_2\sin\left(x-\frac{c\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)\right)}+\frac{k\left(-k^2+3m+\sqrt{k^4-4mk^2}\right)}{-k^2+3\sqrt{k^4-4mk^2}}$$

$$v_{23}(x,t)=\frac{\frac{A1\left(-\mathrm{I}\eta_2\sin\left(x-\frac{c\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)+\eta_2\cos\left(x-\frac{c\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)\right)}{4\left(\mathrm{I}\eta_2\cos\left(x-\frac{c\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)+\eta_2\sin\left(x-\frac{c\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)\right)}+\frac{k\left(-k^2+3m+\sqrt{k^4-4mk^2}\right)}{-k^2+3\sqrt{k^4-4mk^2}}}{\sqrt{\frac{1}{m-k+1}}}$$

$$u_{24}(x,t)=\frac{\left(\mathrm{I}k+\sqrt{-k^2+4m}\right)\cos\left(\frac{c\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-x\alpha}{\alpha}\right)+\left(\mathrm{I}\sqrt{-k^2+4m}-k\right)\sin\left(\frac{c\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-x\alpha}{\alpha}\right)}{2\mathrm{I}\cos\left(\frac{c\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-x\alpha}{\alpha}\right)-2\sin\left(\frac{c\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-x\alpha}{\alpha}\right)}$$

$$v_{24}(x,t)=\frac{\left(\mathrm{I}k+\sqrt{-k^2+4m}\right)\cos\left(\frac{c\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-x\alpha}{\alpha}\right)+\left(\mathrm{I}\sqrt{-k^2+4m}-k\right)\sin\left(\frac{c\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-x\alpha}{\alpha}\right)}{\sqrt{-\frac{1}{-m+k-1}}\left(2\mathrm{I}\cos\left(\frac{c\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-x\alpha}{\alpha}\right)-2\sin\left(\frac{c\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}-x\alpha}{\alpha}\right)\right)}$$

$$u_{25}(x,t)=\frac{\sqrt{2}\eta_1}{\left(x+\frac{\sqrt{2}k\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{2\alpha}\right)\eta_1+\eta_0}+\frac{k}{2},$$

$$v_{25}(x,t)=\frac{\frac{\sqrt{2}\eta_1}{\left(x+\frac{\sqrt{2}k\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{2\alpha}\right)\eta_1+\eta_0}+\frac{k}{2}}{\sqrt{\frac{1}{\frac{1}{4}k^2-k+1}}}$$

$$u_{26}(x,t)=\frac{\sqrt{2}\left(\eta_0\sinh\left(x+\frac{k\sqrt{2}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{2\alpha}\right)-\eta_0\cosh\left(x+\frac{k\sqrt{2}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{2\alpha}\right)\right)}{\eta_0\cosh\left(x+\frac{k\sqrt{2}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{2\alpha}\right)+\eta_1-\eta_0\sinh\left(x+\frac{k\sqrt{2}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{2\alpha}\right)}+\frac{k}{2}+\frac{\sqrt{2}}{2}$$

$$v_{26}(x,t)=\frac{\frac{\sqrt{2}\left(\eta_0\sinh\left(x+\frac{k\sqrt{2}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{2\alpha}\right)-\eta_0\cosh\left(x+\frac{k\sqrt{2}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{2\alpha}\right)\right)}{\eta_0\cosh\left(x+\frac{k\sqrt{2}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{2\alpha}\right)+\eta_1-\eta_0\sinh\left(x+\frac{k\sqrt{2}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{2\alpha}\right)}+\frac{k}{2}+\frac{\sqrt{2}}{2}}{\sqrt{\delta}}$$

$$u_{27}=\sqrt{2}\tanh\left(\frac{\sqrt{2}k\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}+2x\alpha}{2\alpha}\right)+\frac{k}{2}$$

$$v_{27}=\frac{2\sqrt{2}\tanh\left(\frac{\sqrt{2}k\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}+2x\alpha}{2\alpha}\right)+k}{2\sqrt{\delta}}$$

$$u_{28}=\sqrt{2}\tanh\left(\frac{\sqrt{2}k\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}+2x\alpha}{2\alpha}\right)+\frac{k}{2}$$

$$v_{28}=\frac{2\sqrt{2}\tanh\left(\frac{\sqrt{2}k\left(\frac{t\Gamma(\alpha)+1}{\Gamma(\alpha)}\right)^{\alpha}+2x\alpha}{2\alpha}\right)+k}{2\sqrt{\delta}}$$

$$u_{29}(x,t)=\frac{\sqrt{2}\left(\eta_0\sinh\left(x-\frac{c\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)+\eta_0\cosh\left(x-\frac{c\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)\right)}{\eta_0\cosh\left(x-\frac{c\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)+\eta_1+\eta_0\sinh\left(x-\frac{c\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)}$$

$$v_{29}(x,t)=\frac{\sqrt{2}\left(\eta_0\sinh\left(x-\frac{c\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)+\eta_0\cosh\left(x-\frac{c\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)\right)}{\left(\eta_0\cosh\left(x-\frac{c\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)+\eta_1+\eta_0\sinh\left(x-\frac{c\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha}\right)\right)\sqrt{\delta}}$$

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