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On some new matrix transformations

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Abstract

In this paper, we characterize some matrix classes $(\omega(p,s), V_\sigma^\lambda)$, $(\omega_p(s), V_\sigma^\lambda)$ and $(\omega_p(s), V_\sigma^\lambda)_{\text{reg}}$ under appropriate conditions.

1 Introduction

Let w denote the set of all real and complex sequences $x = (x_k)$. By l_∞ and c, we denote the Banach spaces of bounded and convergent sequences $x = (x_k)$ normed by $\|x\| = \sup_k |x_k|$, respectively. A linear functional L on l_∞ is said to be a Banach limit [1] if it has the following properties:

(1) $L(x) \geq 0$ if $n \geq 0$ (*i.e.*, $x_n \geq 0$ for all n),

(2) $L(e) = 1$, where $e = (1, 1, \ldots)$,

(3) $L(Dx) = L(x)$, where the shift operator D is defined by $D(x_n) = \{x_{n+1}\}$.

Let B be the set of all Banach limits on l_∞. A sequence $x \in \ell_\infty$ is said to be almost convergent if all Banach limits of x coincide. Let $\hat{c}$ denote the space of the almost convergent sequences. Lorentz [2] has shown that

$$\hat{c} = \left\{ x \in l_\infty : \lim_m d_{m,n}(x) \text{ exists, uniformly in } n \right\},$$

where

$$d_{m,n}(x) = \frac{x_n + x_{n+1} + x_{n+2} + \cdots + x_{n+m}}{m+1}.$$

The study of regular, conservative, coercive and multiplicative matrices is important in the theory of summability. In [3], King used the concept of the almost convergence of a sequence introduced by Lorentz to define more general classes of matrices than those of regular and conservative ones.

In [4], Schaefer defined the concepts of σ-conservative, σ-regular and σ-coercive matrices and characterized the matrix classes (c, V_σ), $(c, V_\sigma)_{\text{reg}}$ and (l_∞, V_σ), where V_σ denotes the set of all bound sequences, all of whose invariant means (or σ-means) are equal. In [5], Mursaleen characterized the classes $(c(p), V_\sigma)$, $(c(p), V_\sigma)_{\text{reg}}$ and $(l_\infty(p), V_\sigma)$ of matrices, which generalized the results due to Schaefer [4]. In [6], Mohiuddine and Aiyup defined the space $\omega(p,s)$ and obtained necessary and sufficient conditions to characterize the matrices of classes $(\omega(p,s), V_\sigma)$, $(\omega_p(s), V_\sigma)$ and $(\omega_p(s), V_\sigma)_{\text{reg}}$.

Matrix transformations between sequence spaces have also been discussed by Savaş and Mursaleen [7], Basarir and Savaş [8], Mursaleen [5, 9–16], Vatan and Simsek [17], Savaş [18–24], Vatan [25] and many others.

In this paper we characterize the matrix classes from this space to the space V_σ^λ, *i.e.*, we obtain necessary and sufficient conditions to characterize the matrices of classes $(\omega(p,s), V_\sigma^\lambda)$, $(\omega_p(s), V_\sigma^\lambda)$ and $(\omega_p(s), V_\sigma^\lambda)_{\mathrm{reg}}$.

2 Preliminaries

Let σ be a one-to-one mapping from the set N of natural numbers into itself. A continuous linear functional φ on l_∞ is said to be an invariant mean or σ-mean if and only if

(i) $\varphi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all k;
(ii) $\varphi(e) = 1$;
(iii) $\varphi(x) = \varphi(x_{\sigma(k)})$ for all $x \in l_\infty$.

Let V_σ denote the set of bounded sequences all of whose σ-means are equal. We say that a sequence $x = (x_k)$ is σ-convergent if and only if $x \in V_\sigma$. For $\sigma(n) = n + 1$, the set V_σ is reduced to the set $\hat{c}$ of almost convergent sequences [2, 26].

If $x = (x_n)$, write $Tx = (x_{\sigma(n)})$. It is easy to show that

$$V_\sigma = \Big\{x \in l_\infty : \lim_m t_{mn}(x) = L, \text{ uniformly in } n; L = \sigma\text{-}\lim x\Big\},$$

where

$$t_{mn}(x) = \frac{1}{m+1}\sum_{j=0}^{m} T^j x_n$$

and $\sigma^m(n)$ denotes the mth iterate of σ at n.

If p_k is real and positive, we define (see Maddox [27])

$$c_0(p) = \Big\{x : \lim_{k\to\infty} |x_k|^{p_k} = 0\Big\}$$

and

$$c(p) = \Big\{x : \lim_{k\to\infty} |x_k - l|^{p_k} = 0 \text{ for some } l\Big\}.$$

The classes $(\omega(p,s), V_\sigma)$, $(\omega_p(s), V_\sigma)$ and $(\omega_p(s), V_\sigma)_{\mathrm{reg}}$ have been defined by Mohiuddine and Aiyup [6] and, for $p = (p_k)$ with $p_k > 0$, the space $\omega(p,s)$ is defined for $s \geq 0$ by

$$\omega(p,s) = \left\{x : \frac{1}{n}\sum_{k=1}^{n} k^{-s}|x_k - l|^{p_k} \to 0, n \to \infty \text{ for some } l, s \geq 0\right\},$$

where $s = (s_k)$ is an arbitrary sequence with $s_k \neq 0$ $(k = 1, 2, \ldots)$. If $p_k = p$, for each k, we have $\omega(p,s) = \omega_p(s)$.

The sequence space

$$\omega(p) = \left\{x : \frac{1}{n}\sum_{k=1}^{n} |x_k - l|^{p_k} \to 0, n \to \infty\right\}$$

for some l, which has been investigated by Maddox is the special case of $\omega(p,s)$ which corresponds to $s = 0$. Obviously $\omega(p) \subset \omega(p,s)$.

We further define the following.

Let $\lambda = (\lambda_m)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{m+1} \leq \lambda_m + 1, \quad \lambda_1 = 1.$$

A sequence $x = (x_k)$ of real numbers is said to be (σ, λ)-convergent to a number L if and only if $x \in V_\sigma^\lambda$, where

$$V_\sigma^\lambda = \Bigl\{ x \in l_\infty : \lim_{m\to\infty} t_{mn}(x) = L, \text{ uniformly in } n; L = (\sigma,\lambda)\text{-}\lim x \Bigr\},$$

$$t_{mn}(x) = \frac{1}{\lambda_m} \sum_{i \in I_m} x_{\sigma^i(n)},$$

and $I_m = [m - \lambda_m + 1, m]$. Note that $c \subset V_\sigma^\lambda \subset l_\infty$. For $\sigma(n) = n+1$, V_σ^λ reduces to the space $\hat{V}_\lambda$ of almost λ-convergent sequences [28]; and if we take $\sigma(n) = n+1$ and $\lambda_m = m$, then V_σ^λ reduces to $\hat{c}$ (see [29]). Further, if we take $\lambda_m = m$, then V_σ^λ reduces to V_σ.

If E is a subset of ω, then we write E^+ for a generalized Köthe-Toeplitz dual of E; *i.e.*,

$$E^+ = \Bigl\{ a : \sum_k a_k x_k \text{ converges for every } x \in E \Bigr\}.$$

If $0 < p_k \leq 1$, then $\omega^+(p) = \mathbb{M}$, where

$$\mathbb{M} = \Bigl\{ a : \sum_{r=0}^{\infty} \max_r \bigl\{ \bigl(2^r \cdot N^{-1}\bigr)^{\frac{1}{p_k}} |a_k| \bigr\} < \infty \text{ for some integer } N > 1 \Bigr\},$$

and max is the maximum taken over $2^r \leq k < 2^{r+1}$ (see Theorem 4, [30]).

If X is a topological linear space, we denote by X^* the continuous dual of X; *i.e.*, the set of all continuous linear functionals on X. Obviously,

$$\bigl[\omega(p,s)\bigr]^* = \Biggl\{ a : \sum_{r=0}^{\infty} \max_r \Biggl\{ \bigl(2^r \cdot N^{-1}\bigr)^{\frac{1}{p_k}} \biggl| \frac{a_k}{s_k} \biggr| \Biggr\} < \infty \text{ for some integer } N > 1 \Biggr\}.$$

3 Main results

Let X and Y be two nonempty subsets of the space w of complex sequences. Let $A = (a_{nk})$ $(n, k = 1, 2, \ldots)$ be an infinite matrix of complex numbers. We write $Ax = (A_n(x))$ if $A_n(x) := \sum_k a_{nk} x_k$ converges for each n. (Throughout, $\sum_k$ will denote summation over k from $k = 1$ to $k = \infty$.) If $x = (x_k) \in X$ implies that $Ax = (A_n(x)) \in Y$, we say that A defines a (matrix) transformation from X to Y and we denote it by $A : X \to Y$. By (X, Y) we mean the class of matrices A such that $A : X \to Y$.

We now characterize the matrices in the class $(c_0(p), V_{\sigma_0}^\lambda(p))$. We write

$$t_{m,n}(Ax) = \sum_k a(n,k,m) x_k,$$

where

$$a(n,k,m) = \frac{1}{\lambda_m} \sum_{i \in I_m} a_{\sigma^i(n),k}.$$

Theorem 3.1 *Let $0 < p_k \leq 1$, then $A \in (\omega(p,s), V_\sigma^\lambda)$ if and only if*

(i) *there exists an integer $B > 1$ such that for every n*

$$D_n = \sup_m \sum_{r=0}^{\infty} \max_r \left(2^r \cdot B^{-1}\right)^{\frac{1}{p_k}} \left| \frac{a(n,k,m)}{s_k} \right| < \infty;$$

(ii) $\alpha_{(k)} = \{a_{nk}\}_{n=1}^{\infty} \in V_\sigma^\lambda$ *for each k;*

(iii) $\alpha = \{\sum_k a_{nk}\}_{n=1}^{\infty} \in V_\sigma^\lambda$.

In this case the σ-lim of Ax is $(\lim x)[u - \sum_k u_k] + \sum_k u_k x_k$ for every $x \in w(p,s)$, where $u = \sigma\text{-}\lim a$ and $u_k = \sigma\text{-}\lim a_{(k)}$, $k = 1, 2, \ldots$.

Proof Suppose that $A \in (\omega(p,s), V_\sigma^\lambda)$. Define $e^k = (0, 0, \ldots, 1, 0, \ldots)$ having 1 in the kth coordinate sequence. Since e and e^k are in $\omega(p,s)$, necessity of (ii) and (iii) is clear. We know that $\sum_k a(n,k,m)x_k$ converges for each m, n and $x \in \omega(p,s)$. Therefore $(a(n,k,m))_k \in \omega^+(p,s)$ and

$$\sum_{r=0}^{\infty} \max_r \left(2^r \cdot B^{-1}\right)^{\frac{1}{p_k}} \left| \frac{a(n,k,m)}{s_k} \right| < \infty$$

for each m, n (see [31]). Furthermore, if $f_{mn}(x) = t_{mn}(Ax)$, then $\{f_{mn}\}_m$ is a sequence of continuous linear functionals on $\omega(p,s)$ such that $\lim_m t_{mn}(Ax)$ exists. Therefore, by using the Banach-Steinhaus theorem, the necessity of (i) follows immediately.

Conversely, suppose that the conditions (i), (ii) and (iii) hold and $x \in \omega(p,s)$. We know that $(a(n,k,m))_k$ and u_k are in $\omega^+(p,s)$ and that the series $\sum_k a(n,k,m)x_k$ and $\sum_k u_k x_k$ converge for each m, n. Write

$$c(n,k,m) = a(n,k,m) - u_k.$$

Then

$$\sum_k a(n,k,m)x_k = \sum_k u_k x_k + l \sum_k c(n,k,m) + \sum_k c(n,k,m)(x_k - l)$$

by (ii) for some integer $k_0 > 0$, we have

$$\lim_m \sum_{k \leq k_0} c(n,k,m)(x_k - l) = 0, \quad \text{uniformly in } n,$$

where l is the limit of x for $x \in \omega(p,s)$. Since

$$\sup_{m,n} \sum_r \max_r \left(2^r \cdot B^{-1}\right)^{\frac{1}{p_k}} \left|c(n,k,m)\right| \leq 2D_n,$$

$$\lim_m \sum_{k \leq k_0} \left| \frac{a(n,k,m) - u_k}{s_k} \right| \left|s_k(x_k - l)\right| = 0,$$

uniformly in n, whence

$$\lim_n \sum_k a(n,k,m)x_k - l \cdot u + \sum_k u_k(x_k - l).$$

□

Theorem 3.2 *Let $1 \leq p_k < \infty$, then $A \in (\omega_p(s), V_\sigma^\lambda)$ if and only if*

(i) *for every n*

$$M(A) = \sup_m \sum_r 2^{\frac{r}{p}} \left(\sum_r \left| \frac{a(n,k,m)}{s_k} \right|^q \right)^{\frac{1}{q}} < \infty,$$

where $p^{-1} + q^{-1} = 1$;

(ii) $\alpha_{(k)} \in V_\sigma^\lambda$ *for each k;*

(iii) $\alpha \in V_\sigma^\lambda$.

Proof Assume that the conditions are satisfied and let $x \in \omega_p(s)$. Then

$$\left|t_{mn}(Ax)\right| \leq \sum_{r=0}^{\infty} \sum_r \left| \frac{a(n,k,m)s_k x_k}{s_k} \right| \leq \sum_{r=0}^{\infty} \left(\sum_r \left| \frac{a(n,k,m)}{s_k} \right|^q \right)^{\frac{1}{q}} \cdot \left(\sum_r |x_k|^p \right)^{\frac{1}{p}},$$

and hence $t_{mn}(Ax)$ is absolutely and uniformly convergent for each m, n. Note that (i) and (ii) imply that

$$\sum_{r=0}^{\infty} 2^{\frac{r}{p}} \left(\sum_r |s_k u_k| \right)^{\frac{1}{q}} \leq M(A) < \infty,$$

so that by Hölder's inequality, $\sum_k u_k x_k < \infty$. Now as in the converse part of Theorem 3.1, it follows that $A \in (\omega_p(s), V_\sigma^\lambda)$.

Conversely, suppose that $A \in (\omega_p(s), V_\sigma^\lambda)$. Since e^k and e are in $\omega_p(s)$, the necessity of (ii) and (iii) is clear. For the necessity of (i), suppose that

$$t_{mn}(Ax) = \sum_k a(n,k,m)x_k$$

exits for each n whenever $x \in \omega_p(s)$. Then, for each n and $r \geq 0$, write

$$f_{nr}(x) = \sum_r a(n,k,m)x_k.$$

Then $\{f_{nr}\}_m$ is a sequence of continuous linear functionals on $\omega_p(s)$. Since

$$\left|f_{nr}(x)\right| \leq \left(\sum_r \left| \frac{a(n,k,m)}{s_k} \right|^q \right)^{\frac{1}{q}} \cdot \left(\sum_r |s_k \cdot x_k|^p \right)^{\frac{1}{p}} \leq 2^{\frac{r}{p}} \left(\sum_r \left| \frac{a(n,k,m)}{s_k} \right|^q \right)^{\frac{1}{q}} \cdot \|x\|,$$

it follows that for each n,

$$\lim_j \sum_{r=0}^{j} f_{mr}(x) = t_{mn}(Ax)$$

is in the dual space ω_p^*. Hence there exists a K_{mn} such that

$$\left| \frac{a(n,k,m)}{s_k} \right| \leq K_{mn}\|x\|. \tag{3.1}$$

For each n and any integer $j > 0$, define $x \in \omega_p(s)$ as in [30] (Theorem 7, p.173), we get

$$\sum_{r=0}^{j} 2^{\frac{r}{p}} \left(\sum_{r} \left| \frac{a(n,k,m)}{s_k} \right|^q \right)^{\frac{1}{q}} \le K_{mn}.$$

Hence, for each n,

$$\sum_{r=0}^{\infty} 2^{\frac{r}{p}} \left(\sum_{r} \left| \frac{a(n,k,m)}{s_k} \right|^q \right)^{\frac{1}{q}} \le K_{mn} < \infty. \tag{3.2}$$

Since $t_{mn}(Ax)$ is absolutely convergent, we get

$$\left| t_{mn}(Ax) \right| \le \sum_{r=0}^{\infty} 2^{\frac{r}{p}} \left(\sum_{r} \left| \frac{a(n,k,m)}{s_k} \right|^q \right)^{\frac{1}{q}} \|x\|,$$

so that

$$K_{mn} \le \sum_{r=0}^{\infty} 2^{\frac{r}{p}} \left(\sum_{r} \left| \frac{a(n,k,m)}{s_k} \right|^q \right)^{\frac{1}{q}}. \tag{3.3}$$

By virtue of (3.2) and (3.3),

$$K_{mn} = \sum_{r=0}^{\infty} 2^{\frac{r}{p}} \left(\sum_{r} \left| \frac{a(n,k,m)}{s_k} \right|^q \right)^{\frac{1}{q}}.$$

Finally, by (Theorem 11, [30], p.114) for every n, the existence of $\lim_m t_{mn}(Ax)$ on $\omega_p(s)$ implies that

$$\sup_m K_{mn} = \sup_m \sum_{r=0}^{\infty} 2^{\frac{r}{p}} \left(\sum_{r} \left| \frac{a(n,k,m)}{s_k} \right|^q \right)^{\frac{1}{q}} < \infty,$$

which is (i). □

Theorem 3.3 *Let* $0 < p_k < \infty$, *then* $A \in (\omega_p(s), V_\sigma)_{\text{reg}}$ *if and only if conditions* (i), (ii) *with* σ-lim = 0 *and* (iii) *with* σ-lim = +1 *of Theorem* 3.2 *hold*.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors completed the paper together. Both authors read and approved the final manuscript.

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