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Results in Mathematics

Tauberian Theorems for Sequences in 2-Normed Spaces

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Abstract. In this paper we initiate the study of Tauberian theorems for sequences in 2-normed spaces. We introduce a necessary and sufficient Tauberian condition for $(C, 1)$ summable sequences in such spaces. Corollaries allow this condition to be replaced by 2-normed analogues of Schmidt-type slow oscillation condition or Hardy-type two-sided condition.

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1. Introduction and Preliminaries

The concept of 2-normed spaces was introduced by Gähler [8] in 1964. He constructed a real valued function of two variables over a vector space which satisfies certain axioms such that the vector space equipped with this more general structure becomes a locally convex topological vector space. This notion received the attention of a wider audience after the study of White [23] in 1969. In the mentioned study, he defined convergent sequences and Cauchy sequences in 2-normed spaces which lead the birth of theory of 2-Banach spaces.

In the last decade, 2-normed spaces have attained noticeable importance and popularity from the researchers studying on the summability theory. Firstly, Gürdal and Pehlivan [9,10] presented the statistical convergence of sequences in 2-normed spaces. Meanwhile, Şahiner et al. [13] defined the ideal convergence of sequences in such spaces. Later, Savaş [14,15] and Das [5] determined certain new sequence spaces using ideal convergence and Orlicz functions in 2-normed spaces and examined some of their properties. Also, several results on statistical convergence and new generalized statistical convergence definitions such as lacunary statistical convergence and A-statistical

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convergence can be found in Belen and Yildirim [1], Cakalli and Ersan [2] and Dutta and Reddy [7].

Now, we give some basic definitions and notations that are needed in the sequel.

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A mapping $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ satisfying the following four properties

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent;
(ii) $\|x, y\| = \|y, x\|$ for all $x, y \in X$;
(iii) $\|\alpha x, y\| = |\alpha|\|x, y\|$ for all $x, y \in X$ and $\alpha \in \mathbb{R}$;
(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for all $x, y, z \in X$,

is called a 2-norm on X, and the pair $(X, \|\cdot,\cdot\|)$ is called a 2-normed space. In addition, using the above properties it is easy to obtain that $\|x, y\| \geq 0$ and $\|x, y + \alpha x\| = \|x, y\|$ for all $x, y \in X$ and $\alpha \in \mathbb{R}$.

A trivial example of a 2-normed space is $X = \mathbb{R}^2$ equipped with the 2-norm $\|x, y\| :=$ area of the parallelogram determined by the vectors x and y as the adjacent sides, which can be given explicitly by

$$\|x, y\| = |x_1 y_2 - x_2 y_1|, \text{ where } x = (x_1, x_2) \text{ and } y = (y_1, y_2).$$

For $X = \mathbb{R}^3$, we may define the 2-norm

$$\|x, y\| = \max\{|x_1 y_2 - x_2 y_1|, |x_1 y_3 - x_3 y_1|, |x_2 y_3 - x_3 y_2|\},$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Then $(\mathbb{R}^3, \|\cdot,\cdot\|)$ is a 2-normed space.

A sequence (x_n) in a 2-normed space X is said to converge to $L \in X$ (in 2-norm) and written $x_n \xrightarrow{\|\cdot,\cdot\|_X} L$, if

$$\lim_{n\to\infty} \|x_n - L, y\| = 0$$

for all $y \in X$. Moreover, (x_n) is said to be bounded (in 2-norm) and written $x_n = O(1)$ if

$$\|x_n, y\| \leq M \quad (n = 1, 2, \dots)$$

for all $y \in X$. Here and in the sequel, M denotes a positive constant possibly different at each occurrence.

For a given (x_n), the sequence (σ_n) of its arithmetic (also called $(C, 1)$) means is defined by

$$\sigma_n = \frac{1}{n} \sum_{k=1}^{n} x_k \quad (n = 1, 2, \dots).$$

We say that a sequence (x_n) in a 2-normed space X is $(C, 1)$ summable to $L \in X$ and write $x_n \xrightarrow{\|\cdot,\cdot\|_X} L(C, 1)$, if

$$\lim_{n\to\infty} \|\sigma_n - L, y\| = 0 \tag{1}$$

for all $y \in X$.

The following theorem shows that convergence of a sequence in X always implies its $(C,1)$ summability to the same value.

Theorem 1. *If a sequence $(x_n) \in X$ is convergent to $L \in X$, then the sequence (σ_n) of its arithmetic means is convergent to L.*

Proof. Suppose convergence of (x_n) to L, then for each $\epsilon > 0$ and $y \in X$ there exists $n_0 \in \mathbb{N}$ such that $\|x_n - L, y\| \leq \epsilon/2$ if $n > n_0$ and there exists $M > 0$ such that $\|x_n - L, y\| \leq M$ if $n \leq n_0$. So, for all $y \in X$ we have

$$\begin{aligned}
\|\sigma_n - L, y\| &= \left\| \frac{1}{n}\sum_{k=1}^{n} x_k - \frac{1}{n}\sum_{k=1}^{n} L, y \right\| \\
&= \left\| \frac{1}{n}\sum_{k=1}^{n} (x_k - L), y \right\| \\
&\leq \frac{1}{n}\sum_{k=1}^{n} \|x_k - L, y\| \\
&= \frac{1}{n}\sum_{k=1}^{n_0} \|x_k - L, y\| + \frac{1}{n}\sum_{k=n_0+1}^{n} \|x_k - L, y\| \\
&\leq \frac{n_0 M}{n} + \frac{\epsilon}{2}.
\end{aligned}$$

Since $\lim_{n\to\infty} \frac{n_0 M}{n} = 0$, there exists $n_1 \in \mathbb{N}$ such that $\left|\frac{n_0 M}{n}\right| \leq \frac{\epsilon}{2}$ whenever $n > n_1$. Therefore, there exists $N = \max\{n_0, n_1\}$ such that $\|\sigma_n - L, y\| \leq \epsilon$ if $n > N$. This completes the proof. □

Example 1. Let $X = \mathbb{R}^3$ be equipped with the 2-norm given explicitly by the formula

$$\|x, y\| = \max\{|x_1y_2 - x_2y_1|, |x_1y_3 - x_3y_1|, |x_2y_3 - x_3y_2|\}$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$.
Define the sequence $(x_n) \in X$ by

$$x_n = \left(1 + (-1)^n, \frac{1}{2} + \frac{(-1)^{n+1}}{2}, 1\right),$$

and let $y = (y_1, y_2, y_3)$. Then, the sequence (σ_n) of its arithmetic means of (x_n) can be given by

$$\sigma_n = \begin{cases} \left(\dfrac{n-1}{n}, \dfrac{n+1}{2n}, 1\right), & \text{if } n \text{ is odd} \\ \left(1, \dfrac{1}{2}, 1\right), & \text{if } n \text{ is even.} \end{cases}$$

If n is even $(\sigma_n) = (1, 1/2, 1)$ and if n is odd for all $y = (y_1, y_2, y_3) \in \mathbb{R}^3$

$$\begin{aligned}
\lim_{n\to\infty} \|\sigma_n - L, y\| &= \lim_{n\to\infty} \left\| \left(\frac{n-1}{n}, \frac{n+1}{2n}, 1 \right) - \left(1, \frac{1}{2}, 1\right), (y_1, y_2, y_3) \right\| \\
&= \lim_{n\to\infty} \left\| \left(\frac{-1}{n}, \frac{1}{2n}, 0 \right), (y_1, y_2, y_3) \right\| \\
&= \lim_{n\to\infty} \max \left\{ \left| \frac{-y_2}{n} - \frac{y_1}{2n} \right|, \left| \frac{-y_3}{n} \right|, \left| \frac{y_3}{2n} \right| \right\} \\
&= 0.
\end{aligned}$$

Thus, by (1) we conclude that (x_n) is $(C, 1)$ summable to $(1, 1/2, 1)$.

On the other hand, if (x_n) is convergent, the limit should be $(1, 1/2, 1)$. However, since for $y = (0, 1, 1) \in \mathbb{R}^3$

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - L, y\| &= \lim_{n\to\infty} \left\| \left((-1)^n, \frac{(-1)^{n+1}}{2}, 0 \right), (y_1, y_2, y_3) \right\| \\
&= \lim_{n\to\infty} \max \left\{ \left| (-1)^n y_2 - \frac{(-1)^{n+1} y_1}{2} \right|, |(-1)^n y_3|, \left| \frac{(-1)^{n+1} y_3}{2} \right| \right\} \\
&= \lim_{n\to\infty} \max \left\{ |(-1)^n|, |(-1)^n|, \left| \frac{(-1)^{n+1}}{2} \right| \right\} \\
&= 1 \neq 0,
\end{aligned}$$

the sequence (x_n) is not convergent.

Notice that the converse holds only under some suitable condition(s). Such condition(s) is called Tauberian condition(s) with respect to the summability method in question and the resulting theorem is said to be a Tauberian theorem. The name of these theorems goes back to Tauber [21], who was the first to prove theorems of this type.

Although several research papers have been written by a number of authors dealing with the summability theory in 2-normed spaces, yet Tauberian theorems for sequences in such spaces remains unctouched. In the present paper, to fill this gap we initiate the study of Tauberian theorems for sequences in 2-normed spaces.

2. Tauberian Theorems for the $(C, 1)$ Summability

In this section we present our main Tauberian theorem for the $(C, 1)$ summability. Also we establish two classical Tauberian theorems in 2-normed spaces as corollaries to our main theorem. For the sake of completeness of the paper we give proofs of theorems in the following section.

Let (p_n) and (q_n) be increasing sequences of positive integers such that $p_n < q_n$ and $p_n \to \infty$ as $n \to \infty$. We write $(p, q) \in \Lambda$ if the condition

$$\liminf_{n\to\infty} \frac{q_n}{p_n} > 1 \tag{2}$$

holds.

Theorem 2. *Let (x_n) be a sequence in a 2-normed space $(X, ||\cdot,\cdot||)$. If (x_n) is $(C,1)$ summable to $L \in X$, then (x_n) converges to L if and only if*

$$\inf_{(p,q)\in\Lambda} \limsup_{n\to\infty} \left\| \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} (x_k - x_n), y \right\| = 0 \tag{3}$$

for all $y \in X$.

A few remarks are appropriate here.

Remark 1. Consider the special case of (3) for $p_n = n$ and $q_n = [\lambda n]$ with $\lambda > 1$, where $[\,\cdot\,]$ means the integer part. In this case, the condition (3) reduces to

$$\inf_{\lambda>1} \limsup_{n\to\infty} \left\| \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} (x_k - x_n), y \right\| = 0. \tag{4}$$

Also, for $0 < \lambda < 1$, Móricz [12] gave the following condition

$$\inf_{0<\lambda<1} \limsup_{n\to\infty} \left\| \frac{1}{n - [\lambda n]} \sum_{k=[\lambda n]+1}^{n} (x_n - x_k), y \right\| = 0. \tag{5}$$

By using restrictions identical to (4) and (5), Móricz [12] proved a Tauberian theorem for the $(C,1)$ summability of sequences of complex numbers.

Remark 2. Following Schmidt [16] (see also Stanojević [18]), we say that a sequence $(x_n) \in X$ is slowly oscillating in 2-norm if

$$\inf_{\lambda>1} \limsup_{n\to\infty} \max_{n<k\le[\lambda n]} \|x_k - x_n, y\| = 0 \tag{6}$$

for all $y \in X$, or equivalently,

$$\inf_{0<\lambda<1} \limsup_{n\to\infty} \max_{[\lambda n]<k\le n} \|x_n - x_k, y\| = 0.$$

It is plain that (6) is satisfied if and only if

$$\|x_k - x_n, y\| \to 0 \quad \text{whenever} \quad y \in X \text{ and } 1 < \frac{k}{n} \to 1 \ \ (k, n \to \infty). \tag{7}$$

Using $\epsilon > 0$ and $\lambda = \lambda(\epsilon) > 1$, (7) may be written as follows:
To every $\epsilon > 0$ there exists $N = N(\epsilon)$ and $\lambda = \lambda(\epsilon) > 1$, as close to 1 as we wish, such that

$$\|x_k - x_n, y\| \le \epsilon \tag{8}$$

whenever $y \in X$ and $N \le n < k \le \lambda n$.

If (x_n) is slowly oscillating in 2−norm, then conditions (4) and (5) are satisfied. In recent years, slow oscillation condition were used in a number of

Tauberian theorems for real or complex sequences [4,17,19], improper integrals [3,22] and sequences of fuzzy numbers [20,24].

Remark 3. If the two-sided condition of Hardy-type [11], $n\Delta x_n = O(1)$, where $\Delta x_n = x_n - x_{n-1}$, holds then (x_n) is slowly oscillating in 2-norm.

Considering the above Remarks, we obtain 2-normed analogues of classical Tauberian theorems due to Schmidt [16] and Hardy [11], respectively.

Corollary 1. *Let $(x_n) \in X$ be $(C, 1)$ summable to $L \in X$. If (x_n) is slowly oscillating in $2-$norm, then (x_n) converges to L.*

Corollary 2. *Let $(x_n) \in X$ be $(C, 1)$ summable to $L \in X$. If*

$$n\Delta x_n = O(1), \tag{9}$$

then (x_n) converges to L.

3. Proofs of Main Results

The following lemmas play basic roles in the proof of our main theorem.

Lemma 1. *Let (p_n) be an increasing sequence of positive integers such that $p_n \to \infty$ as $n \to \infty$. If the sequence $(x_n) \in X$ is $(C, 1)$ summable to $L \in X$, then (σ_{p_n}) converges to L.*

Proof. Assume (x_n) is $(C, 1)$ summable to L. That is $\sigma_n \xrightarrow{\|\cdot,\cdot\|_X} L$. Then, since (σ_{p_n}) is a subsequence of (σ_n), it follows $\sigma_{p_n} \xrightarrow{\|\cdot,\cdot\|_X} L$. □

Lemma 2. *Let (x_n) be a sequence in a 2-normed space $(X, \|\cdot,\cdot\|)$ which is $(C, 1)$ summable to $L \in X$. If $(p, q) \in \Lambda$, then the sequence*

$$(D_{p,q})_n = \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} x_k$$

also converges to L.

Proof. We verify this lemma by showing that the following 2-norm tends to 0 as $n \to \infty$.

$$\begin{aligned}
\|(D_{p,q})_n - L, y\| &= \|(D_{p,q})_n + \sigma_{q_n} - \sigma_{q_n} - L, y\| \\
&= \left\| \frac{1}{q_n - p_n} \sum_{k=1}^{q_n} x_k - \frac{1}{q_n - p_n} \sum_{k=1}^{p_n} x_k + \frac{1}{q_n} \sum_{k=1}^{q_n} x_k - \frac{1}{q_n} \sum_{k=1}^{q_n} x_k - L, y \right\| \\
&= \left\| \frac{p_n}{q_n - p_n} \frac{1}{q_n} \sum_{k=1}^{q_n} x_k - \frac{p_n}{q_n - p_n} \frac{1}{p_n} \sum_{k=1}^{p_n} x_k + \frac{1}{q_n} \sum_{k=1}^{q_n} x_k - L, y \right\| \\
&\leq \frac{p_n}{q_n - p_n} \|\sigma_{q_n} - \sigma_{p_n}, y\| + \|\sigma_{q_n} - L, y\|.
\end{aligned}$$

Hence we attain

$$\|(D_{p,q})_n - L, y\| \leq \frac{p_n}{q_n - p_n} \|\sigma_{q_n} - \sigma_{p_n}, y\| + \|\sigma_{q_n} - L, y\| . \tag{10}$$

Besides, by (2) we have

$$\limsup_{n\to\infty} \frac{p_n}{q_n - p_n} = \left(\liminf_{n\to\infty} \frac{q_n}{p_n} - 1 \right)^{-1} < \infty.$$

Therefore the proof follows from Lemma 1, (10) and the $(C,1)$ summability of (x_n) to L. □

Proof of Theorem 2. Necessity. Suppose (x_n) is convergent to L, whence the convergence of (σ_n) to L follows. Consider arbitrary sequences (p_n) and (q_n) such that $(p,q) \in \Lambda$ to satisfy (3). Thus by Lemma 2, for all $y \in X$ we get

$$\begin{aligned} &\lim_{n\to\infty} \left\| \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} (x_k - x_n), y \right\| \\ &\leq \lim_{n\to\infty} \left\| \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} x_k - L, y \right\| + \lim_{n\to\infty} \|x_n - L, y\| = 0; \end{aligned}$$

that is, we get an even stronger conclusion than (3).
Sufficiency. Now, assume that (3) holds. Then, for any given $\epsilon > 0$, there exist sequences (p_n) and (q_n) satisfying $(p,q) \in \Lambda$ such that

$$\limsup_{n\to\infty} \left\| \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} (x_k - x_n), y \right\| \leq \epsilon \tag{11}$$

for all $y \in X$. By (11), the assumed $(C,1)$ summability of (x_n), and Lemma 2, we conclude

$$\begin{aligned} \limsup_{n\to\infty} \|x_n - L, y\| \leq \limsup_{n\to\infty} &\left\| \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} x_k - L, y \right\| \\ + \limsup_{n\to\infty} &\left\| \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} (x_k - x_n), y \right\| \leq \epsilon \end{aligned}$$

for all $y \in X$. Since $\epsilon > 0$ is arbitrary, the convergence of (x_n) to L follows.

Proof of Corollary 1. Let (x_n) be slowly oscillating in 2-norm. So, we get

$$\begin{aligned} \left\| \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} (x_k - x_n), y \right\| &\leq \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} \|x_k - x_n, y\| \\ &\leq \max_{n < k \leq [\lambda n]} \|x_k - x_n, y\| . \end{aligned}$$

Taking the lim sup of the inequality above as $n \to \infty$,

$$\limsup_{n\to\infty} \left\| \frac{1}{[\lambda n]-n} \sum_{k=n+1}^{[\lambda n]} (x_k - x_n), y \right\| \le \limsup_{n\to\infty} \max_{n<k\le[\lambda n]} \|x_k - x_n, y\| . \quad (12)$$

Now, considering the inf of both sides of the (12) for $\lambda > 1$ we conclude

$$\inf_{\lambda>1} \limsup_{n\to\infty} \left\| \frac{1}{[\lambda n]-n} \sum_{k=n+1}^{[\lambda n]} (x_k - x_n), y \right\| = 0 \quad (13)$$

for each $y \in X$. Note that (13) is the special case of the condition (3) in Theorem 2 (see Remark 1). Therefore, the proof is completed.

Proof of Corollary 2. Suppose (9) holds. Then $\|n\Delta x_n, y\| \le M$ for every $y \in X$ and some positive M. Hence, slow oscillation of (x_n) follows.

$$\begin{aligned} \|x_k - x_n, y\| &= \left\| \sum_{j=n+1}^{k} (x_j - x_{j-1}), y \right\| = \left\| \sum_{j=n+1}^{k} \Delta x_j, y \right\| \le \sum_{j=n+1}^{k} \|\Delta x_j, y\| \\ &\le \sum_{j=n+1}^{k} \frac{M}{j} \le \sum_{j=n+1}^{k} \frac{M}{n} = \left(\frac{k}{n} - 1\right) M \to 0 \ \text{ as } \ \frac{k}{n} \to 1. \end{aligned}$$

Now, the proof is completed by Corollary 1.

4. Extensions of Classical Tauberian Theorems to (H, k) Summability

In the present section, at first, we remind an obvious generalization of the $(C, 1)$ summability which is known as Hölder or (H, k) summability. Later, we extend our classical Tauberian results in Sect. 2 to (H, k) method of summability.

A sequence (x_n) may not be summable $(C, 1)$, but the sequence (σ_n) may be summable $(C, 1)$, in other words, the iteration of the $(C, 1)$ method may produce a convergent sequence. k-fold application of $(C, 1)$ method gives the sequence $(\sigma_n^{(k)})$ as follows:

$$\sigma_n^{(k)} = \begin{cases} \dfrac{1}{n} \displaystyle\sum_{j=1}^{n} \sigma_j^{(k-1)}, & k \ge 1 \\ x_n, & k = 0. \end{cases}$$

Here, if (x_n) is transformed into a convergent sequence after k-fold iteration process, that is, if

$$\lim_{n\to\infty} \left\| \sigma_n^{(k)} - L, y \right\| = 0 \quad (14)$$

for all $y \in X$, then we say that (x_n) is (H, k) summable to L, and write $x_n \xrightarrow{\|\cdot,\cdot\|_X} L(H, k)$.

Note that $(H,0)$ summability is the ordinary convergence in 2-norm, and the $(H,1)$ method of summability is equivalent to the $(C,1)$ summability.

It is worth mentioning that, trivially, (H,k) summability of a sequence implies its $(H,k+1)$ summability to the same value where $k \geq 0$. However, the converse is not necessarily true as the following example shows.

Example 2. Let $X = \mathbb{R}^2$ be equipped with the 2-norm given explicitly by the formula

$$\|x,y\| = |x_1y_2 - x_2y_1| \text{ where } x = (x_1,x_2), \text{ and } y = (y_1,y_2).$$

Consider the sequence $(x_n) \in X$, defined by

$$x_n = \left(2(-1)^{n+1}n + (-1)^n, 0\right),$$

and let $y = (y_1,y_2)$. Taking the first arithmetic mean of (x_n) gives the sequence

$$\sigma_n = \left((-1)^{n+1}, 0\right).$$

Now, applying the same averaging process to the sequence (σ_n) we get

$$\sigma_n^{(2)} = \begin{cases} \left(\dfrac{1}{n}, 0\right), & \text{if } n \text{ is odd} \\ (0,0), & \text{if } n \text{ is even.} \end{cases}$$

Therefore, by (14) we obtain that (x_n) is $(H,2)$ summable to $(0,0)$, but not summable $(H,1)$.

We require the next lemma for the proof of the Theorem 3. The proof of this lemma is inspired by Dik [6, p. 59].

Lemma 3. *If $(x_n) \in X$ is slowly oscillating in 2-norm, then*

$$\left\|\frac{1}{n}\sum_{k=1}^{n} k\Delta x_k, y\right\| \leq M$$

for all $y \in X$.

Proof. Suppose that (x_n) is slowly oscillating in 2-norm. For $\lambda > 1$ we define

$$w_n(x,\lambda) = \max_{n<k\leq[\lambda n]} \left\|\sum_{j=n+1}^{k} \Delta x_j, y\right\|.$$

Hence we obtain

$$\begin{aligned}\left\|\sum_{k=1}^{n} k\Delta x_k, y\right\| &= \left\|\sum_{j=0}^{\infty} \sum_{\frac{n}{2^{j+1}}\leq k<\frac{n}{2^j}} k\Delta x_k, y\right\| \\ &\leq \sum_{j=0}^{\infty}\left\|\sum_{\frac{n}{2^{j+1}}\leq k<\frac{n}{2^j}} k\Delta x_k, y\right\| \leq \left(\sum_{j=0}^{\infty}\frac{n}{2^j}\right) w_{\left[\frac{n}{2^{j+1}}\right]}(x,\lambda) \leq nM,\end{aligned}$$

which completes the proof. $\square$

Theorem 3. *Let $(x_n) \in X$ be (H,k) summable to $L \in X$. If (x_n) is slowly oscillating in $2-$norm, then (x_n) converges to L.*

Proof. First, we prove that if (x_n) is slowly oscillating in 2-norm, then so is the sequence (σ_n). To this end, let $\epsilon > 0$ be given. Then, there exists $N = N(\epsilon)$ and $\lambda = \lambda(\epsilon) > 1$ such that (8) is satisfied.

Let $N \le n < k \le \lambda n$. By the definition of $(C,1)$ summability we obtain

$$\begin{aligned}
\|\sigma_k - \sigma_n, y\| &= \left\| \frac{1}{k}\sum_{j=1}^{k} x_j - \frac{1}{n}\sum_{j=1}^{n} x_j, y \right\| \\
&= \left\| \frac{n-k}{kn}\sum_{j=1}^{n} x_j + \frac{1}{k}\sum_{j=n+1}^{k} x_j, y \right\| \\
&= \left\| \frac{k-n}{kn}\sum_{j=1}^{n} (x_n - x_j) + \frac{1}{k}\sum_{j=n+1}^{k} (x_j - x_n), y \right\| \\
&= \left\| \frac{k-n}{kn}\sum_{j=1}^{n}\sum_{i=j+1}^{n} \Delta x_i + \frac{1}{k}\sum_{j=n+1}^{k} (x_j - x_n), y \right\| \\
&\le \frac{k-n}{kn}\left\| \sum_{i=2}^{n} (i-1)\Delta x_i, y \right\| + \frac{1}{k}\sum_{j=n+1}^{k} \left\|(x_j - x_n), y\right\|.
\end{aligned}$$

Using Lemma 3 and slow oscillation definition of (x_n) in (8) we get

$$\|\sigma_k - \sigma_n, y\| = \left(1 - \frac{n}{k}\right)(M+\epsilon) < (\lambda - 1)(M+\epsilon) < \epsilon$$

whenever $y \in X$ and $N \le n < k \le \lambda n$ provided that $1 < \lambda < (1+\epsilon)/(M+\epsilon)$. That is, we get the slow oscillation of (σ_n). Applying same procedure above, we obtain that

$$(\sigma_n^{(m)}) \text{ is slowly oscillating for all integer } m \ge 0. \tag{15}$$

Furthermore, by the assumption since

$$\sigma_n^{(k)} \xrightarrow{\|\cdot,\cdot\|_X} L, \tag{16}$$

we have $\sigma_n^{(k-1)} \xrightarrow{\|\cdot,\cdot\|_X} L(H,1)$. Then, taking $m = k-1$ in (15) and applying Corollary 1 to the sequence $(\sigma_n^{(k-1)})$ yields

$$\sigma_n^{(k-1)} \xrightarrow{\|\cdot,\cdot\|_X} L. \tag{17}$$

Now, considering (16) and (17) and continuing in the same fashion, we conclude

$$\sigma_n \xrightarrow{\|\cdot,\cdot\|_X} L.$$

Consequently, the proof follows from Corollary 1. □

Theorem 4. *Let $(x_n) \in X$ be (H,k) summable to $L \in X$. If*

$$n\Delta x_n = O(1), \tag{18}$$

then (x_n) converges to L.

Proof. Assume (18) holds, then by the definition $\|n\Delta x_n, y\| \leq M$ for each $y \in X$ and $M > 0$. Besides, we have

$$n\Delta x_n = O(1) \;\Rightarrow\; n\Delta\sigma_n = O(1). \tag{19}$$

Indeed,

$$\begin{aligned}
\|n\Delta\sigma_n, y\| &= n\left\|\frac{1}{n}\sum_{k=1}^{n} x_k - \frac{1}{n-1}\sum_{k=1}^{n-1} x_k, y\right\| \\
&= \frac{1}{n-1}\left\|(n-1)x_n - \sum_{k=1}^{n-1} x_k, y\right\| \\
&= \frac{1}{n-1}\left\|\sum_{k=1}^{n-1}(x_n - x_k), y\right\| \\
&= \frac{1}{n-1}\left\|\sum_{k=1}^{n-1}\sum_{j=k+1}^{n} \Delta x_j, y\right\| \\
&= \frac{1}{n-1}\left\|\sum_{j=2}^{n}(j-1)\Delta x_j, y\right\| \\
&\leq \frac{1}{n-1}\sum_{j=2}^{n}\|j\Delta x_j, y\| \leq M
\end{aligned}$$

for each $y \in X$ and some $M > 0$. Taking (19) into account we find that

$$n\Delta\sigma_n^{(m)} = O(1) \;\text{ for each integer } m \geq 0. \tag{20}$$

On the other hand, (H,k) summability of (x_n) to L implies

$$\sigma_n^{(k-1)} \xrightarrow{\|\cdot,\cdot\|_X} L(H,1). \tag{21}$$

Choosing $m = k-1$ in (20) and considering (21), we get $(H, k-1)$ summability of (x_n) to L by using Corollary 2. This also implies

$$\sigma_n^{(k-2)} \xrightarrow{\|\cdot,\cdot\|_X} L(H,1). \tag{22}$$

Taking (21) and (22) into account and continuing in this way, we conclude

$$s_n \xrightarrow{\|\cdot,\cdot\|_X} L(H,1).$$

Therefore, the proof follows from Corollary 2. □

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