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Universidad Católica del Norte
Antofagasta - Chile

On graded pseudo 2-prime ideals

Anass Assarrar
University S. M. Ben Abdellah Fez, Morocco
Najib Mahdou
University S. M. Ben Abdellah Fez, Morocco
Muroj Al-Shboul
Yarmouk University, Jordan
Ünsal Tekir
Marmara University, Turkey
Svetlin Georgiev Georgiev
University of Sofia, Bulgaria
Suat Koç
Istanbul Medeniyet University, Turkey
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Abstract

In this paper, we study graded pseudo 2-prime ideals of graded commutative rings with nonzero identities. Let G be a commutative additive monoid with an identity element 0, and $R = \bigoplus_{g \in G} R_g$ be a commutative graded ring with a nonzero identity element. A proper graded ideal I of R is said to be a graded pseudo 2-prime ideal if whenever $ab \in I$ for some homogeneous elements $a, b \in R$, then $a^{2n} \in I^n$ or $b^{2n} \in I^n$ for some $n \in \mathbf{N}$. Besides giving many properties of graded pseudo 2-prime ideals, we characterize graded almost valuation domains in terms of our new concept.

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1. Introduction

In this article, all rings under consideration are assumed to be commutative with nonzero identity and all modules are assumed to be nonzero unital. Let R will always represent such a ring, M will represent such an R-module. Also, G will represent a commutative additive monoid with identity element 0 unless otherwise stated. A ring R is said to be a *graded ring* (or *G-graded ring*) if $R = \bigoplus_{g\in G} R_g$, where R_g's are additive subgroups of R, and $R_gR_h \subseteq R_{g+h}$ for all $g, h \in G$. The elements of R_g are called *homogeneous elements of degree g*, and also the set of all homogeneous elements of R is denoted by $h(R) = \bigcup_{g\in G} R_g$. An ideal I of R is said to be a *graded ideal* (or sometimes called *homogeneous ideal*) if $I = \bigoplus_{g\in G}(I \cap R_g)$, or equivalently, $a = \sum_{g\in G} a_g \in I$ implies that $a_g \in I$ for all $g \in G$, where $a \in R$. Our aim in this paper is to introduce a new class of graded ideals in graded commutative rings and use them to characterize graded almost valuation domains firstly defined by Bakkari et al. in [3]. The notion of prime ideals and its generalizations have a vital role in Commutative Algebra since they have some applications to other areas such as Graph Theory, General Topology, Cryptology, Algebraic Geometry and etc. See, for example, [4]-[18]. A proper graded ideal I of R is said to be a *graded prime ideal* if whenever $ab \in I$ for some $a, b \in h(R)$ then $a \in I$ or $b \in I$. Let G be a group with identity 0. For any graded ideal I of R, the graded radical of I is defined as:

$$Gr(I) = \{a = \sum_{g\in G} a_g \in R : \forall g \in G, \exists\ n_g > 0 \text{ such that } a_g^{n_g} \in I\}.$$

Note that $Gr(I)$ is a graded ideal of R [25]. Recently, there have been various generalizations of graded prime ideals in several papers. Among the many recent generalizations of the notion of graded prime ideals in the literature, we find the following, due to Al-Zoubi et al [1]. A proper graded ideal I of R is said to be a *graded 2-absorbing ideal* if whenever $abc \in I$ for some $a, b, c \in h(R)$ then $ab \in I$ or $ac \in I$ or $bc \in I$. Also, following [6], a proper graded ideal I of R is said to be a *graded 2-prime ideal* if whenever $ab \in I$ for some $a, b \in h(R)$ then $a^2 \in I$ or $b^2 \in I$. We call a proper graded ideal I of R as a *graded pseudo 2-prime ideal* if whenever $ab \in I$ for some $a, b \in h(R)$ then there exists $n \in \mathbf{N}$ such that $a^{2n} \in I^n$ or $b^{2n} \in I^n$. Among other things in this paper, we give the relations between graded pseudo 2-prime ideals and other classical graded ideals such as graded 2-prime ideals, graded 2-absorbing ideals and graded irreducible ideals (See, Proposition 1, Example 2, Example 3, Theorem 1, Proposition 2). Also, we investigate the stability of graded pseudo 2-prime ideals under graded homomorphism, in factor ring, in localization of graded rings, in cartesian product of graded rings, in graded trivial extension (See, Proposition 3, Corollary 1, Proposition 4, Theorem 3, Theorem 4, Theorem 6). Finally, we study finite union of graded pseudo 2-prime ideals and prove the graded pseudo 2-prime avoidance theorem (See, Theorem 7 and Theorem 8).

2. Preliminaries

We devote this section to some conventions and a recall of some terminology. Unless otherwise stated, G will denote a commutative additive monoid with an identity element denoted by 0. Let R be a G-graded ring and I be a graded ideal of R. Then the factor ring R/I is a G-graded ring by $(R/I)_g = (I + R_g)/I$ for all $g \in G$. Suppose that A, B are G-graded rings and $R = A \times B$. Then R is a G-graded ring by $R_g = A_g \times B_g$ for all $g \in G$. Also, it can be easily seen that an ideal I of R is a graded ideal if and only if $I = J \times K$ for some graded ideals J of A and K of B. The following theorem is well known, but we give it here for the sake of completeness.

Theorem 1. *Let R be a G-graded ring and I, J graded ideals of R and $x \in h(R)$. Then, the following statements are satisfied.*
(1) $I + J$, IJ and $I \cap J$ are graded ideals of R.
(2) If x has a cancellable degree, $(I : x) = \{a \in R : ax \in I\}$ is a graded ideal of R.
(3) $Rx = \{ax : a \in R\}$ is a graded ideal of R.

Let G be a group, R be a G-graded ring, and $S \subseteq h(R)$ be a multiplicatively closed set. Then $S^{-1}R$ is a G-graded ring by $\left(S^{-1}R\right)_g = \left\{\frac{a}{s} : a \in R_h,\ s \in S \cap R_{h-g}\right\}$ for all $g \in G$. If I is a graded ideal of R, then it can be easily seen that $S^{-1}I$ is a graded ideal of $S^{-1}R$.

Suppose that R is a G-graded ring, where G is a commutative additive monoid with an identity element 0, and M is an R-module. Then M is said to be a G-graded R-module if whenever $M = \bigoplus_{g\in G} M_g$ for some additive subgroups M_g of M such that $R_g M_h \subseteq M_{g+h}$ for all $g, h \in G$. The set of homogeneous elements of M of degree g is denoted by M_g and the set of all homogeneous element of M is denoted by $h(M) = \bigcup_{g\in G} M_g$. An R-submodule N of M is said to be a graded submodule if whenever

$m = \sum_{g\in G} m_g \in N$ then $m_g \in N$ for all $g \in G$. Let R be a G-graded ring and M be a G-graded R-module. The graded trivial extension of R-module M is denoted by $RM = R \oplus M$ with componentwise addition and the following multiplication $(a,m)(b,m') = (ab, am' + bm)$ for all $a, b \in R$ and $m, m' \in M$. Then RM is a G-graded ring by $(RM)_g = R_g \oplus M_g$ for all $g \in G$ [19]. Suppose that I is an ideal of R and N is a submodule of M. Then $IN(= I \oplus N$ additively) is a graded ideal of RM if and only if I is a graded ideal of R, N is a graded submodule of M and $IM \subseteq N$ [19, Theorem 2.4]. For more informations and other terminology on graded rings and modules, we refer [20] to the reader.

3. Graded pseudo 2-prime ideals

Recall from [22] that a proper ideal I of R is said to be a pseudo 2-prime ideal if whenever $ab \in I$ for some $a, b \in R$ then there exists $n \in \mathbf{N}$ such that $a^{2n} \in I^n$ or $b^{2n} \in I^n$. Now, we extend the notion of pseudo 2-prime ideals to the context of graded rings.

Definition 1. *Let I be a proper graded ideal of R. Then I is said to be a graded pseudo 2-prime ideal if whenever $ab \in I$ for some homogeneous elements $a, b \in R$, then $a^{2n} \in I^n$ or $b^{2n} \in I^n$ for some $n \in \mathbf{N}$.*

By definition, one can easily see that every pseudo 2-prime graded ideal of R is also a graded pseudo 2-prime ideal. However, the converse is not always true. See, the following example.

Example 1. *Consider $R = \mathbf{Z}[i]$ and $G = \mathbf{Z}_2$. Then R is G-graded by $R_0 = \mathbf{Z}$ and $R_1 = i\mathbf{Z}$. Consider the graded ideal $I = 5R$ of R. We show that I is a graded prime ideal of R. Let $xy \in I$ for some $x, y \in h(R)$.*
<u>Case (1)</u>: Let $x, y \in R_0$. In this case, $x, y \in \mathbf{Z}$ such that 5 divides xy, and then either 5 divides x or 5 divides y as 5 is a prime number, which implies that either $x \in I$ or $y \in I$.
<u>Case (2)</u>: Let $x, y \in R_1$. In this case, $x = ia$ and $y = ib$ for some $a, b \in \mathbf{Z}$ such that 5 divides $xy = -ab$, and then 5 divides ab in $\mathbf{Z}$, and again either 5 divides a or 5 divides b, which implies that either 5 divides $x = ia$ or 5 divides $y = ib$, and hence either $x \in I$ or $y \in I$.
<u>Case (3)</u>: Let $x \in R_0$ and $y \in R_1$. In this case, $x \in \mathbf{Z}$ and $y = ib$ for some $b \in \mathbf{Z}$ such that 5 divides $xy = ixb$ in R, that is $ixb = 5(\alpha + i\beta)$ for some $\alpha, \beta \in \mathbf{Z}$, which gives that $xb = 5\beta$, that is 5 divides xb in $\mathbf{Z}$, and again

either 5 divides x or 5 divides b, and then either 5 divides x or 5 divides $y = ib$ in R. Thus, we have that $x \in I$ or $y \in I$.

In other cases, one can similarly show that $x \in I$ or $y \in I$. So, I is a graded prime ideal of R, and then I is a graded pseudo 2-prime ideal of R. On the other hand, I is not a pseudo 2-prime ideal of R since $(2-i)(2+i) \in I$, $(2-i)^{2n} \notin I^n$ and $(2+i)^{2n} \notin I^n$ for each positive integer n.

Proposition 1. *Let R be a G-graded ring and I a proper graded ideal of R. Then we have*
(1) If I is a graded 2-prime ideal of R, then it is a graded pseudo 2-prime ideal of R.
(2) If G is a group and I is a graded pseudo 2-prime ideal of R, then $Gr(I)$ is a graded prime ideal of R.

Proof. (1) Suppose that I is a graded 2-prime ideal of R and $ab \in I$ for some $a, b \in h(R)$. This gives $a^2 \in I$ or $b^2 \in I$. Then we have $a^{2n} \in I^n$ or $b^{2n} \in I^n$ as needed.
(2) Let $ab \in Gr(I)$ for some $a, b \in h(R)$. Then there exists $m \in \mathbf{N}$ such that $(ab)^m = a^m b^m \in I$. Since $a^m, b^m \in h(R)$ and I is a graded pseudo 2-prime ideal, we conclude that $a^{2mn} \in I^n$ or $b^{2mn} \in I^n$ for some $n \in \mathbf{N}$. This implies that $a \in Gr(I)$ or $b \in Gr(I)$, which completes the proof. □

The converses of previous proposition (1) and (2) need not to be true. See the following example.

Example 2. *Let k be an arbitrary field and $R = k[X, Y]$, where X, Y are indeterminates over k. Then $R = \cdots \bigoplus 0 \bigoplus k \bigoplus (kX + kY) \bigoplus (kX^2 + kXY + kY^2) \bigoplus \cdots$ is a $\mathbf{Z}$-graded ring. Now, put $P = (X^8, XY, Y^8)$. Consider $A = R/P$ and $I = (X^4, XY, Y^4)/P$. Then note that A is a $\mathbf{Z}$-graded ring and I is a proper graded ideal of A. Also, it is clear that $Gr(I) = (X, Y)/P$ and $Gr(I)^8 = (\overline{0})$, where $\overline{0} = 0 + P$. Then I is not a graded 2-prime ideal since $\overline{X}.\overline{Y} \in I$ but $\overline{X}^2, \overline{Y}^2 \notin I$. On the other hand, it is clear that I is a graded pseudo 2-prime ideal since $Gr(I)^8 = (\overline{0})$ and $Gr(I)$ is a graded prime ideal of R.*

Example 3. *Let R be the $\mathbf{Z}$-graded ring as in the previous example and $I = (X^m, XY, Y^m)$ where $m > 2$ is an integer. Then I is not a graded pseudo 2-prime ideal since $XY \in I$ but $X^{2n}, Y^{2n} \notin I^n$ for each $n \in \mathbf{N}$. However, $Gr(I) = (X, Y)$ is a graded maximal ideal of R.*

From above observations, we have the following diagram which explain the place of graded pseudo 2-prime ideal in the set of all graded ideals of an arbitrary graded ring R. Recall that a proper graded ideal I of a G-graded ring R is said to be a graded quasi primary ideal if $ab \in I$ for $a, b \in h(R)$ implies that $a \in Gr(I)$ or $b \in Gr(I)$.

$$\boxed{\text{graded 2-prime ideal}} \longrightarrow \boxed{\text{graded pseudo 2-prime ideal}} \longrightarrow \boxed{\text{graded quasi primary ideal}}$$

Figure 1: graded pseudo 2-prime ideal vs other classical graded ideals

The concept of graded primary ideals has been introduced and examined in [24]. A proper graded ideal I of R is said to be graded primary if whenever $xy \in I$ for some $x, y \in h(R)$, then either $x \in I$ or $y \in Grad(I)$. The authors in [21], gave a generalization of graded primary ideals; a proper graded ideal I of R is said to be graded 2-absorbing primary if whenever $xyz \in I$ for some $x, y, z \in h(R)$, then $xy \in I$, $xz \in Grad(I)$ or $yz \in Grad(I)$.

Theorem 1. *Let R be a G-graded ring and I a proper graded ideal of R. The following statements are satisfied.*
(1) If I is a graded pseudo 2-prime ideal of R, then I is a graded 2-absorbing primary ideal of R.
(2) If $Gr(I)^2 \subseteq I$, then I is a graded 2-prime ideal of $R \Leftrightarrow$ I is a graded pseudo 2-prime ideal of $R \Leftrightarrow$ $Gr(I)$ is a graded prime ideal of R.
(3) Suppose that I is a graded 2-absorbing ideal of R. Then I is a graded 2-prime ideal of R $\Leftrightarrow$ I is a graded pseudo 2-prime ideal of R.
(4) The zero ideal is a graded pseudo 2-prime ideal of R $\Leftrightarrow$ (0) is a graded prime ideal of R.

Proof. (1) Let $abc \in I$ for some $a, b, c \in h(R)$. This gives $(ac)^{2n} \in I^n \subseteq I$ or $b^{2n} \in I^n \subseteq I$ for some $n \in \mathbf{N}$. This completes the proof.
(2) The implications "I is a graded 2-prime ideal of $R \Rightarrow I$ is a graded pseudo 2-prime ideal of $R \Rightarrow$ $Gr(I)$ is a graded prime ideal of R" follows from Proposition 1. Let $Gr(I)$ be a graded prime ideal of R and $Gr(I)^2 \subseteq I$. Choose $a, b \in h(R)$ such that $ab \in I \subseteq Gr(I)$. Since $Gr(I)$ is a graded prime ideal, we conclude that $a \in Gr(I)$ or $b \in Gr(I)$. Since $Gr(I)^2 \subseteq I$, which implies that $a^2 \in I$ or $b^2 \in I$. Therefore, $I$is a graded 2-prime ideal of R.

(3) Let I be a graded 2-absorbing ideal and graded pseudo 2-prime ideal. Choose $a, b \in h(R)$ such that $ab \in I$. This gives $a^{2n} \in I^n$ or $b^{2n} \in I^n$. Which implies that $a \in Gr(I)$ or $b \in Gr(I)$. Then by [1, Lemma 2.5], $a^2 \in I$ or $b^2 \in I$ since I is a graded 2-absorbing ideal of R. The rest follows from Proposition 1.
(4) Suppose that $Gr(0)$ is a graded prime ideal of R and $ab = 0$ for some $a, b \in h(R)$. This gives $a \in Gr(0)$ or $b \in Gr(0)$. Then we have $a^{2n} = 0 \in (0)^n$ or $b^{2n} = 0 \in (0)^n$ for some $n \in \mathbf{N}$. Hence, the zero ideal is a graded pseudo 2-prime ideal. The rest follows from Proposition 1. □

Remark 1. *In Example 3, I is not a graded pseudo 2-prime ideal of R. On the other hand, $Grad(I)$ is a graded maximal ideal of R, so by ([24], Proposition 1.11), I is a graded primary ideal of R, and hence I is a graded 2-absorbing primary ideal of R. Therefore, the converse of Corollary 1 (1) is not necessarily true.*

The next two examples show that graded primary ideals and graded pseudo 2-prime ideals are completely different:

Example 4. *In Example 3, I is not a graded pseudo 2-prime ideal of R. On the other hand, $Grad(I)$ is a graded maximal ideal of R, so by ([24], Proposition 1.11), P is a graded primary ideal of R.*

Example 5. *Consider $R = k[X, Y]$, where k is a field, and $G = \mathbf{Z}$. Then R is G-graded by $R_n = \bigoplus_{i+j=n, i,j\geq 0} kX^iY^j$ for all $n \in \mathbf{Z}$. Consider the graded ideal $I = (X^2, XY)$ of R. Then $Grad(I) = (X)$ is a graded prime ideal of R and $(Grad(I))^2 = (X^2) \subseteq I$. So, it is clear that I is a graded pseudo 2-prime ideal of R. On the other hand, I is not a graded primary ideal of R since $X, Y \in h(R)$ such that $XY \in I$, $X \notin I$ and $Y \notin Grad(I)$.*

Recall from [2] that a graded ideal I of R is said to be a *graded irreducible ideal* if whenever $I = I_1 \cap I_2$ for some graded ideals I_1 and I_2 of R, then $I = I_1$ or $I = I_2$. Now, we give the relations between graded pseudo 2-prime ideals and graded irreducible ideals.

Proposition 2. *Let I be a graded proper ideal of a graded ring R. Assume that I^n is a graded irreducible ideal for some $n \in \mathbf{N}$ and $(I^n : x^{2n}) = (I^n : x^{2n-1})$ for each $x \in R - I$. Then I is a graded pseudo 2-prime ideal of R.*

Proof. Let $ab \in I$ for some $a, b \in h(R)$. If a or b is in I, then $a^{2n} \in I^n$ or $b^{2n} \in I^n$ which completes the proof. Now, assume that $a, b \in R - I$. Then by assumption, $(I^n : a^{2n}) = (I^n : a^{2n-1})$ and $(I^n : b^{2n}) = (I^n : b^{2n-1})$. Let $x \in (I^n + Ra^{2n}) \cap (I^n + Rb^{2n})$ be a homogeneous element. Then $x = i_1 + r_1a^{2n} = i_2 + r_2b^{2n}$ for some $i_1, i_2 \in I^n$ and $r_1, r_2 \in R$. Since $xb^{2n} = i_1b^{2n} + r_1(ab)^{2n} = i_2b^{2n} + r_2b^{4n} \in I^n$, we have $r_2b^{4n} \in I^n$. This implies that $r_2b^{2n} \in (I^n : b^{2n}) = (I^n : b^{2n-1})$, and thus we have $r_2b^{4n-1} \in I^n$. By continuing in this manner, we obtain $r_2b^{2n} \in I^n$. Thus we conclude that $x = i_2 + r_2b^{2n} \in I^n$. This implies that $I^n = (I^n + Ra^{2n}) \cap (I^n + Rb^{2n})$. Since I^n is a graded irreducible ideal, we have $I^n = I^n + Ra^{2n}$ or $I^n = I^n + Rb^{2n}$. Thus, we have $a^{2n} \in I^n$ or $b^{2n} \in I^n$. □

Let R, S be two graded rings and $f : R \to S$ be a ring homomorphism. f is said to be a *graded ring homomorphism* of degree h, where $h \in G$, if $f(R_g) \subseteq R_{g+h}$ for all $g \in G$. Also, f is said to be a graded ring homomorphism if it is a graded ring homomorphism of degree h for some $h \in G$.

Proposition 3. *Suppose that $f : R \to S$ is a graded epimorphism and I is a proper graded ideal of R. The following statements are satisfied.*
(1) If I is a graded pseudo 2-prime ideal of R containing $Ker(f)$, then $f(I)$ is a graded pseudo 2-prime ideal of S.
(2) If $f(I)$ is a graded pseudo 2-prime ideal of S such that $Ker(f) \subseteq I^n$ for all $n \in \mathbf{N}$,then I is a graded pseudo 2-prime ideal of R.

Proof. (1) Let $xy \in f(I)$ for some homogeneous elements $x, y \in S$. Since f is a surjective graded ring homomorphism, there exist homogeneous elements $a, b \in R$ such that $x = f(a), y = f(b)$. This gives $f(ab) \in f(I)$. As $Ker(f) \subseteq I$, we conclude that $ab \in I$. Since I is a graded pseudo 2-prime ideal of R, we have $a^{2n} \in I^n$ or $b^{2n} \in I^n$. Thus we have $x^{2n} = f(a)^{2n} = f(a^{2n}) \in f(I^n) = f(I)^n$ or $y^{2n} \in f(I)^n$. Therefore, $f(I)$ is a graded pseudo 2-prime ideal of S.
(2) Let $ab \in I$ for some $a, b \in h(R)$. Then we have $f(a)f(b) \in f(I)$. Since $f(I)$ is a graded pseudo 2-prime ideal of S, we conclude that $f(a)^{2n} = f(a^{2n}) \in f(I)^n = f(I^n)$ or $f(b^{2n}) \in f(I^n)$. As $I^n \supseteq Ker(f)$, we have $a^{2n} \in I^n$ or $b^{2n} \in I^n$. Hence, I is a graded pseudo 2-prime ideal of R. □

As an immediate consequence of previous proposition, we have the following explicit result.

Corollary 1. *Let $J \subseteq I$ be two proper graded ideals of R. The following statements are satisfied.*

(1) If I is a graded pseudo 2-prime ideal of R, then I/J is a graded pseudo 2-prime ideal of R/J.

(2) If I/J is a graded pseudo 2-prime ideal of R/J and $J \subseteq I^n$ for each $n \in \mathbf{N}$, then I is a graded pseudo 2-prime ideal of R.

The conditions "$Ker(f) \subseteq I^n$" and "$J \subseteq I^n$" in Proposition 3 (ii) and Corollary 1 (ii) are necessary. See the following example.

Example 6. *Let R be the $\mathbf{Z}$-graded ring as in Example 2 and $I = J = (X^4, XY, Y^4)$. Then note that $Gr(I/J) = Gr(I)/J = (X, Y)/J$ is a graded prime ideal of R. Thus, by Theorem 1 (iv), I/J is a graded pseudo 2-prime ideal of R/J. On the other hand, note that JI^n for all $n \geq 2$. However, by Example 3, we know that I is not a graded pseudo 2-prime ideal of R.*

Let P be an ideal of R. The set $\{x \in R : xy \in P \text{ for some } y \notin P\}$ is denoted by $Z_R(P)$. In fact, if $P = (0)$, then $Z_P(R)$ is the set of all zero divisors of R and we use $zd(R)$ instead of $Z_P(R)$. Also, the set of all regular elements (i.e., non zero divisors) of R will be designated by $reg(R) = A - zd(R)$.

Proposition 4. *Let R be a graded ring and $S \subseteq h(R)$ be a multiplicatively closed set of R. The following statements are satisfied.*
(1) If I is a graded pseudo 2-prime ideal of R and $S \cap I = \emptyset$, then $S^{-1}I$ is a graded pseudo 2-prime ideal of $S^{-1}R$.
(2) If $S^{-1}I$ is a graded pseudo 2-prime ideal of $S^{-1}R$ and $S \cap Z_R(I^n) = \emptyset$ for each $n \in \mathbf{N}$,then I is a graded pseudo 2-prime ideal of R.

Proof. (1) Let $\frac{a}{s}\frac{b}{t} = \frac{ab}{st} \in S^{-1}I$ for some $a, b \in h(R)$ and $s, t \in S$. Then we conclude that $uab \in I$ for some $u \in S \subseteq h(R)$. Since I is a graded pseudo 2-prime ideal, we have $a^{2n} \in I^n$ or $(ub)^{2n} = u^{2n}b^{2n} \in I^n$ which gives $\left(\frac{a}{s}\right)^{2n} = \frac{a^{2n}}{s^{2n}} \in S^{-1}I^n = (S^{-1}I)^n$ or $\left(\frac{b}{t}\right)^{2n} = \frac{u^{2n}b^{2n}}{u^{2n}t^{2n}} \in S^{-1}I^n = (S^{-1}I)^n$. Hence, $S^{-1}I$ is a graded pseudo 2-prime ideal of $S^{-1}R$.
(2) Let $ab \in I$ for some homogeneous elements $a, b \in R$. Then we have $\frac{a}{1}\frac{b}{1} = \frac{ab}{1} \in S^{-1}I$. As $S^{-1}I$ is a graded pseudo 2-prime ideal of $S^{-1}R$, we have $\left(\frac{a}{1}\right)^{2n} = \frac{a^{2n}}{1} \in (S^{-1}I)^n = S^{-1}I^n$ or $\frac{b^{2n}}{1} \in S^{-1}I^n$. Then there exist $s \in S$ such that $sa^{2n} \in I^n$ or $sb^{2n} \in I^n$. Now, assume that $sa^{2n} \in I^n$. In this case, we clearly have $a^{2n} \in I^n$ since $S \cap Z_R(I^n) = \emptyset$. In other case, we get $b^{2n} \in I^n$ which completes the proof. □

The condition "$S \cap Z_R(I^n) = \emptyset$ for each $n \in \mathbf{N}$" in Proposition 4 can not be removed. See the following example.

Example 7. *Let $R = \mathbf{Z}$ be a graded ring with trivial graduation and $I = 10\mathbf{Z}$. Now, put $S = R - 5\mathbf{Z}$. Then $S^{-1}I = S^{-1}(5\mathbf{Z})$ is a graded prime ideal of $S^{-1}R$ so it is a graded pseudo 2-prime ideal. On the other hand, note that $2^n \in S \cap Z_R(I^n)$ for all $n \in \mathbf{N}$ since $2^n \cdot 5^n \in I^n$ and $5^n \notin I^n$. However, I is not a graded pseudo 2-prime ideal since $2 \cdot 5 \in I$, $2^{2n} \notin I^n$ and $5^{2n} \notin I^n$ for all $n \in \mathbf{N}$.*

Recall from that [23], a G-graded ring R, where G is a group, is said to be a graded *von Neumann regular ring* if for each $a \in R_g$ $(g \in G)$, there exists $x \in R_{g^{-1}}$ such that $a = a^2x$. Also, a graded commutative ring R with unity is said to be a *graded field* if every nonzero homogeneous element of R is unit [26]. Clearly, every field is a graded field, however, the converse is not necessarily true, see ([26], Example 3.6). By ([23], Example 3.3), every graded field is a graded von Neumann regular ring. In the same context, a graded commutative ring R with unity is said to be a *graded domain* if R has no homogeneous zero divisor. Obviously, every domain is a graded domain, however, the converse is not necessarily true, see ([26], Example 3.6). In fact, we have the following:

Proposition 5. *Let R be a graded domain where G is a group. Then R is a graded field if and only if R is a graded von Neumann regular ring.*

Proof. Suppose that R is a graded von Neumann regular ring. Let $0 \neq a \in h(R)$. Then $a \in R_g$ for some $g \in G$, and then $a = a^2x$ for some $x \in R_{g^{-1}}$, which yields that $a(1 - ax) = 0$. Note that, $ax \in R_g R_{g^{-1}} \subseteq R_e$ and $1 \in R_e$, so $1 - ax \in R_e \subseteq h(R)$. Since R is a graded domain and $a \neq 0$, we have $1 - ax = 0$, that is $1 = ax$, which means that a is a unit. Therefore, R is a graded field. The converse follows from ([23], Example 3.3). □

Lemma 1. *Let G be a group, R be a graded ring and I be a graded ideal of R. If R is a graded von Neumann regular ring, then R/I is a graded von Neumann regular ring.*

Proof. Let $a + I \in (R/I)_g$ for some $g \in G$. Then $a \in R_g$, and then $a = a^2x$ for some $x \in R_{g^{-1}}$, which yields that $x + I \in (R/I)_{g^{-1}}$ such that $(a + I)^2(x + I) = (a^2 + I)(x + I) = a^2x + I = a + I$. Therefore, R/I is a graded von Neumann regular ring. □

Theorem 2. *Let G be a group, R be a graded von Neumann regular ring and I be a graded ideal of R. Then I is a graded prime ideal of R if and only if I is a graded maximal ideal of R.*

Proof. Suppose that I is a graded prime ideal of R. Then R/I is a graded domain. On the other hand, R/I is a graded von Neumann regular ring by Lemma 1. So, R/I is a graded field by Proposition 5, and hence I is a graded maximal ideal of R. The converse is clear. □

By ([23], Lemma 3.7), if R is a graded von Neumann regular ring, where G is a group and $x \in h(R)$, then $Rx = Ra$ for some idempotent element $a \in R_e$. So, $(Rx)^n = Rx$ for each positive integer n, and then $Grad(Rx) = Rx$. Now, we introduce the following:

Proposition 6. *Let G be a group, R be a graded von Neumann regular ring and consider the graded ideal $I = Rx$ for some $x \in h(R)$. Then the following statements are equivalent:*

1. *I is a graded maximal ideal of R.*
2. *I is a graded prime ideal of R.*
3. *I is a graded primary ideal of R.*
4. *I is a graded 2-prime ideal of R.*
5. *I is a graded pseudo 2-prime ideal of R.*
6. *$Grad(I)$ is a graded prime ideal of R.*

Proof. (1) ⇒ (2) and (2) ⇒ (3) are clear.

(3) ⇒ (4) follows from $Grad(I) = I$.

(4) ⇒ (5) follows from Proposition 1 (1).

(5) ⇒ (6) follows from Theorem 1 (2).

(6) ⇒ (1) follows from $Grad(I) = I$ and from Theorem 2. □

Theorem 3. *Let A, B be two graded commutative rings and I, J be two graded ideals of A and B, respectively. Suppose that $R = A \times B$ and $P = I \times J$. The following statements are equivalent.*

(1) P is a graded pseudo 2-prime ideal of R.

(2) $I = A$ and J is a graded pseudo 2-prime ideal of B or I is a graded pseudo 2-prime ideal of A and $J = B$.

Proof. (1) $\Rightarrow$ (2) : Suppose that P is a graded pseudo 2-prime ideal of R. Since $(1,0)(0,1) = (0,0) \in P$, we conclude that $(1,0)^{2n} = (1,0) \in P^n \subseteq P$ or $(0,1)^{2n} = (0,1) \in P^n \subseteq P$. This implies that $I = A$ or $J = B$. Without loss of generality, we may assume that $I = A$ and $J \neq B$. Now, we will show that J is a graded pseudo 2-prime ideal of B. To see this, take some homogeneous elements $c, d \in B$ such that $cd \in J$. Then we have $(0,c)(0,d) = (0,cd) \in P$. As P is a graded pseudo 2-prime ideal, we have $(0,c)^{2n} = (0,c^{2n}) \in P^n$ or $(0,d)^{2n} = (0,d^{2n}) \in P^n$, which implies that $c^{2n} \in J^n$ or $d^{2n} \in J^n$. Thus, J is a graded pseudo 2-prime ideal of B.
(2) $\Rightarrow$ (1) : Let $J = B$ and I is a graded pseudo 2-prime ideal of A. Suppose that $(a,b)(c,d) \in P$ for some $(a,b),(c,d) \in h(R)$. Then we have $ac \in I$ for some homogeneous elements $a, c \in A$. As I is a graded pseudo 2-prime ideal, we conclude that $a^{2n} \in I^n$ or $c^{2n} \in I^n$. This gives $(a,b)^{2n} \in P^n$ or $(c,d)^{2n} \in P^n$. Therefore, P is a graded pseudo 2-prime ideal of R. In other case, one can similarly prove the claim. □

By using previous result and induction on n, we have the following explicit result.

Theorem 4. *Let A_i be a commutative graded ring and I_i be a graded ideal of A_i for each $i = 1, 2, \ldots, n$, where $n \in \mathbf{N}$. Suppose that $A = A_1 \times A_2 \times \cdots \times A_n$ and $I = I_1 \times I_2 \times \cdots \times I_n$. The following statements are equivalent.*
(1) I is a graded pseudo 2-prime ideal of A.
(2) I_i is a graded pseudo 2-prime ideal of A_i for some $i \in \{1, 2, \ldots, n\}$ and $I_k = A_k$ for each $k \in \{1, 2, \ldots, n\} - \{i\}$.

Let G be a torsionless grading monoid, $R = \bigoplus_{g \in G} R_g$ be a G-graded integral domain, and H be the saturated multiplicative set of nonzero homogeneous elements of R, i.e $H = \cup_{g \in G} R_g \setminus \{0\}$. Then $R_H = \oplus_{g \in \langle G \rangle} (R_H)_g$ with $(R_H)_g = \{\frac{a}{b} \mid a \in A_{g+h}, 0 \neq b \in R_h\}$ and $\langle G \rangle$ is the group ring of G, i.e $\langle G \rangle = \{a - b : a, b \in G\}$. Then R is said to be a graded almost valuation domain if for every nonzero homogeneous element $x \in R_H$, there exists an integer $n = n(x) > 1$ such that $x^n \in R$ or $x^{-n} \in R$, or equivalently, for every nonzero homogeneous elements $a, b \in R$, there exists an integer $n > 1$ such that $a^n | b^n$ or $b^n | a^n$ [3]. Now, we will characterize graded almost valuation domains in terms of graded pseudo 2-prime ideals.

Theorem 5. *Let G be a torsionless grading monoid and $R = \bigoplus_{g \in G} R_g$ be a G-graded integral domain. The following statements are equivalent.*

(1) R is a graded almost valuation domain.
(2) Every proper graded ideal is a graded pseudo 2-prime ideal of R.
(3) Every proper principal graded ideal is a graded pseudo 2-prime ideal of R.

Proof. (1) $\Rightarrow$ (2) : Suppose that I is a proper graded ideal of R and $ab \in I$ for some homogeneous elements $a, b \in R$. We may assume that a, b are nonzero. Since R is a graded almost valuation domain, there exists $n \in \mathbf{N}$ such that $a^n | b^n$ or $b^n | a^n$. Assume that $a^n | b^n$. Then we can write $b^n = a^n t$ for some $t \in R$. This gives $b^{2n} = (ab)^n t \in I^n$. In the other case, we have $a^{2n} \in I^n$. Thus, I is a graded pseudo 2-prime ideal of R.
(2) $\Rightarrow$ (3) : It is clear.
(3) $\Rightarrow$ (1) : Let $a, b \in h(R)$ be two nonzero elements of R. If a or b is a unit, then there is nothing to prove. So assume that a, b are nonunits. Now, put $I = (ab)$. Then by (iii), we have that I is a graded pseudo 2-prime ideal of R. Since $ab \in I$, there exists $n \in \mathbf{N}$ such that $a^{2n} \in I^n = (a^n b^n)$ or $b^{2n} \in I^n = (a^n b^n)$. Without loss of generality, we may assume that $a^{2n} \in I^n = (a^n b^n)$, that is, $a^{2n} = x a^n b^n$ for some homogeneous element $x \in R$. This gives $a^n(a^n - x b^n) = 0$. As R is a graded integral domain, we have $a^n - x b^n = 0$, which implies that $b^n | a^n$. In the other case, we can similarly show that $a^n | b^n$. Therefore, R is a graded almost valuation domain. □

Theorem 6. *Let R be a G-graded ring and M be a G-graded R-module. Suppose that I is a graded ideal of R and N is a graded submodule of M such that $IM \subseteq N$. The following statements are satisfied.*
(1) If IN is a graded pseudo 2-prime ideal of RM, then I is a graded pseudo 2-prime ideal of R.
(2) If I is a graded pseudo 2-prime ideal of R and there exists $n \in \mathbf{N}$ such that $(I^n : a^{2n}) = (I^n : a^{2n-1})$ for each $a \in R - I$, then IN is a graded pseudo 2-prime ideal of RM.
(3) Suppose that there exists $n \in \mathbf{N}$ such that $(I^n : a^{2n}) = (I^n : a^{2n-1})$ for each $a \in R - I$. Then IN is a graded pseudo 2-prime ideal of RM if and only if I is a graded pseudo 2-prime ideal of R.

Proof. (1) Let $ab \in I$ for some $a, b \in h(R)$. Then we have $(a, 0)(b, 0) = (ab, 0) \in IN$. Since IN is a graded pseudo 2-prime ideal, we have $(a, 0)^{2n} = (a^{2n}, 0) \in (IN)^n$ or $(b, 0)^{2n} = (b^{2n}, 0) \in (IN)^n$. Then, we conclude that $a^{2n} \in I^n$ or $b^{2n} \in I^n$. Hence, I is a graded pseudo 2-prime ideal of R.

(2) Let $(a,m)(b,m') = (ab, am' + bm) \in IN$ for some $(a,m),(b,m') \in h(RM)$. Then $ab \in I$. Since I is a graded pseudo 2-prime ideal of R, we conclude that $a^{2n} \in I^n$ or $b^{2n} \in I^n$. Let $a^{2n} \in I^n$.

Case 1: Assume that $a \in I$. Then we get $(a,m)^{2n} = (a^{2n}, 2na^{2n-1}m) \in I^{2n} I^{2n-1} M \subseteq I^n I^n M \subseteq (IN)^n$, as required.

Case 2: Assume that $a \notin I$. Then by assumption, $(I^n : a^{2n}) = (I^n : a^{2n-1}) = R$, that is, $a^{2n-1} \in I^n$. Then we have $(a,m)^{2n} = (a^{2n}, 2na^{2n-1}m) \in I^n I^n M \subseteq (IN)^n$. Therefore, IN is a graded pseudo 2-prime ideal of RM.
(3) Follows from (1) and (2). □

Let $I, I_1, I_2, \ldots, I_n$ be arbitrary ideals of R and $I \subseteq \bigcup_{i=1}^{n} I_i$. The covering $\bigcup_{i=1}^{n} I_i$ is said to be an efficient covering if none of the I_i's can be removed. Now, we will give the graded pseudo 2-prime avoidance theorem. To prove this, we need the following result.

Theorem 7. *Let R be a G-graded ring and $I \subseteq \bigcup_{i=1}^{n} I_i$ be an efficient covering of graded ideals of R. Suppose that $I(I_k)I \cap Gr(I_m)$ for every $k \neq m$. Then no I_k is a graded pseudo 2-prime ideal of R for each $k \in \{1, 2, \ldots, n\}$.*

Proof. Without loss of generality, we may assume that I_1 is a graded pseudo 2-prime ideal of R. By efficient covering, one can easily prove that $I \cap \bigcap_{i=2}^{n} I_i \subseteq I \cap I_1$. Since $I \cap Gr(I_m)I(I_1)$ for each $m \neq 1$, there exists $x_m \in h(R) \cap Gr(I_m) \cap I \setminus Gr(I_1)$. Then there is $t_m \in \mathbf{N}$ such that $x_m^{t_m} \in I_m \cap I \setminus Gr(I_1)$. Now, put $a = x_2^{t_2}$ and $b = x_3^{t_3} x_4^{t_4} \cdots x_n^{t_n}$. Then we have $ab = x_2^{t_2} x_3^{t_3} x_4^{t_4} \cdots x_n^{t_n} \in I \cap \bigcap_{i=2}^{n} I_i \subseteq I_1$. Since I_1 is a graded pseudo 2-prime ideal, we have $a^{2r} \in I_1^r$ or $b^{2r} \in I_1^r$ for some $r \in \mathbf{N}$. If $a^{2r} \in I_1^r$, then $x_2 \in Gr(I_1)$, which is a contradiction. If $b^{2r} \in I_1^r$, then similarly $x_i \in Gr(I_1)$ for some $i \in \{3, 4, \ldots, n\}$, again a contradiction. Hence, I_1 is not a graded pseudo 2-prime ideal of R. Similarly,I_k is not a graded pseudo 2-prime ideal of R for each $1 \leq k \leq n$. □

Theorem 8. *(Graded Pseudo 2-prime Avoidance Theorem) Let R be a G-graded ring and $I \subseteq \bigcup_{i=1}^{n} I_i$ be a covering of graded ideals of R. Suppose that at most two of I_i's are not graded pseudo 2-prime ideal and $I \cap Gr(I_k)I \cap Gr(I_m)$ for each $k \neq m$. Then $I \subseteq I_i$ for some $i \in \{1, 2, \ldots, n\}$.*

Proof. We may reduce the covering to efficient one. So assume that $I \subseteq \bigcup_{i=1}^{n} I_i$ is an efficient covering. If $n > 2$, then at least one of I_i is a graded pseudo 2-prime ideal, which contradicts with the previous theorem. Thus we have $n \leq 2$. The rest is easy. $\square$

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Anass Assarrar
Modelling and Mathematical Structures Laboratory,
Department of Mathematics,
Faculty of Science and Technology of Fez,
University S. M. Ben Abdellah Fez,
Box 2202,
Morocco
e-mail anass.smb.assarrar@gmail.com
Corresponding author

Najib Mahdou
Modelling and Mathematical Structures Laboratory,
Department of Mathematics,
Faculty of Science and Technology of Fez,
University S. M. Ben Abdellah Fez,
Box 2202,
Morocco
e-mail mahdou@hotmail.com

Muroj Al-Shboul
Department of Mathematics,
Yarmouk University,
Irbid,
Jordan
e-mail: murojshboul@yahoo.com

Ünsal Tekir
Department of Mathematics,
Marmara University,
Istanbul,
Turkey
e-mail: utekir@marmara.edu.tr

and

Suat Koç
Department of Mathematics,
Istanbul Medeniyet University,
Istanbul,
Turkey
e-mail: suat.koc@medeniyet.edu.tr